Recall:

$(G, \cdot)$ group:

- binary operation:

  $$G \times G \to G$$

- associative
- has identity
- inverse

Ex:

- Sym (any object)
- $(\mathbb{Z}/n\mathbb{Z}, +)$
- $n \times n$ invertible matrices
  matrix mult
- (nonzero real numbers, $\cdot$) mult
- $(\mathbb{R}^n, +)$

Groups w/ few elements

- $|G|$ ← size of group $= 1$

  means $G = \{1\}$

- $|G| = 2$, means $G = \{1, g\}$

	1	g
1	1	g
g	g	1

No choice here

$$\begin{bmatrix} 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & \vdots & \ddots & 1 & 0 & 0 \\ 0 & \cdots & & & 1 \end{bmatrix}$$

- $|G| = 3$, $G = \{1, a, b\}$

all elts. of G are power of a.

	1	a	a^2
1	1	a	a^2
a	a	a	1
a^2	a^2	1	a

	1	a	b
1	1	a	b
a	a	b	1
b	b	1	a

Note: $ab \neq 1b \neq b$
$\Rightarrow ab = 1$

no choice here

· $|G| = 4$, $G = \{1, a, b, c\}$

$a \neq ab \neq b \quad (\Rightarrow ab = 1 \text{ or } c)$

case i)

Suppose $ab = c$ (*)

	1	a	b	c
1	1	a	b	c
a	a		c	
b	b	c		
c	c			

1	a	b	c
a	1	c	b
b	c	1	a
c	b	a	1

1	a	b	c
a	b	c	1
b	c	1	a
c	1	a	b

$ab = c$

$a^2 = b$

	1	a	a^2	a^3
1	1	a	a^2	a^3
a	a	a^2	a^3	1
a^2	a^2	a^3	1	a
a^3	a^3	1	a	a^2

$KV_4 = \mathrm{Sym}(\circ\!-\!I\!-\!\circ)$

Case II) $ab = 1$

	1	a	b	c
1	1	a	b	c
a	a	c	1	b
b	b	1	c	a
c	c	b	a	1

$a = a$
$b = a^3$
$c = a^2$

$|G| = 4$ has 2 cases
KV_4 or cyclic group of order 4.

<u>Defn</u>: A group G is <u>cyclic</u>
if one can relabel the elements s.t
the Cayley table becomes

	1	a	a^2	$\cdots$		a^n
1	1	a	a^2			
a	a	a^2				
a^2	a^2	a^3	a^4			
$\vdots$	$\vdots$					
a^n	a^n					$a^{n \times n}$
$\vdots$						

such "a" is a <u>generator</u> of G.

Note: standard ex. of a cyclic group of order n is $\mathbb{Z}/n\mathbb{Z}$.

Non example: KV_4 is not cyclic

<u>Defn</u>: if the Cayley table of G is <u>symmetric</u> then
G is <u>abelian</u>.

Commens: $ab = ba$ for all $a, b \in G$

this will always happen if $G = \mathbb{Z}/n\mathbb{Z}$

or in $(\mathbb{R}\setminus\{0\}, \cdot)$

Not abelian: $\mathrm{Sym}(\Delta)$

- $(n \times n$ invertibles, $\cdot)$

Def^n: The order of G is the # of elts

Def^n: the order of $g \in G$, that is the $\min\{k \in \mathbb{N} \text{ s.t } \underbrace{g \cdot\ \ g}_{k \text{ times}} = 1\}$

If $g^k \neq 1$ for all $k > 0$

we say $\mathrm{order}_G(g) = \infty$

Ex: If $|G| = 1$

$\Rightarrow G = \{1\}$, $\mathrm{ord}(1) = 1$

- $|G| = 2$, $G = \{1, a\}$
  $\mathrm{ord}_G(g) = 2$

- $|G| = 3 \Rightarrow \mathrm{ord}(b) = 3 = \mathrm{ord}(a)$

$$\begin{tikzcd} {|G| = 4} \arrow[r] \arrow[d] & C_4 \\ KV_4 & \end{tikzcd}$$

C_4:

$\text{ord}(a^3) = 4 = \text{ord}(a)$

$\text{ord}(a^2) = 2$

KV_4:

$\text{ord}(a) = 2 = \text{ord}(b) = \text{ord}(c)$

An example of a group where all non-identity elements have ∞ order is $(\mathbb{Z}, +)$