Lec. 5

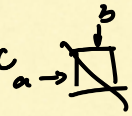
Abelian

order → element
→ group

subgroup

product

automorphism

Defn: group G is Abelian when the cayley table is symmetric 

($\forall a,b \in G$
$ab = ba$)

examples: $(\mathbb{Z}/n\mathbb{Z}, +)$

$\mathrm{sym}(\square)$ $\mathrm{sym}(\text{o-I-o}) = \mathrm{sym}(H) = KV_4$

Non-example: $\mathrm{sym}(\Delta)$ Equilateral triangle

$\ell \cdot a \neq a \cdot \ell$

Ex on orders and subgroups:

Given a group $(G, \cdot)$, a <u>subgroup</u> is a subset $H \subseteq G$ s.t H each multiplication inherited from G, is a group itself.

This entails:

* H needs to contain the identity from G

* the product of any two things in H should be in H.

* Inverses of things in H must be in H.

orders of elements in

KV_4	C_4
$\begin{array}{c\|ccc} 1 & \leftrightarrow & \updownarrow & \circlearrowright \\ \hline \leftrightarrow & 1 & \circlearrowright & \updownarrow \\ \updownarrow & \circlearrowright & 1 & \leftrightarrow \\ \circlearrowright & \updownarrow & \leftrightarrow & 1 \end{array}$	$\begin{array}{c\|ccc} 1 & x & x^2 & x^3 \\ \hline x & x^2 & x^3 & 1 \\ x^2 & x^3 & 1 & x \\ x^3 & 1 & x & x^2 \end{array}$
ord 1: 1 ord 2: $\updownarrow, \circlearrowright, \leftrightarrow$ ord 3: / Done	ord(2) = x^2 ord(3) = / ord(4) = x, x^3 Done

$$\operatorname{ord}(g) = \min \left\{ \begin{matrix} k \in \mathbb{N} \\ k > 0 \end{matrix} \quad \underbrace{g \cdot \ldots g}_{k} = 1 \right.$$

	KV_4	C_4 trivial
Subgroups of order 1	$\{1\}$	$\{1\}$
of order 2	$\{1, \circlearrowright\}, \{1, \leftrightarrow\}$ $\{1, \updownarrow\}$	$\{1, x^2\}$
of order 3	×	×

of order 4 | KV_4 | C_4 trivial

Def^n: An automorphism of a group G is a relabeling of its elements that preserves the cayley table.

Aut $(G) \leftrightarrow$ the collection of these

Ex: $G = \{1\}$

no relabeling except $1 \to 1$.

$\text{Aut}(G) = \{\text{identity relabeling}\}$

$G = (\mathbb{Z}/2\mathbb{Z}, +)$

$\text{Aut}(G) = \{\text{Identity relabeling}\}$

$G = \{\bar{0}, \bar{1}\}$ $\begin{pmatrix} \bar{0} & \bar{1} \\ \bar{1} & \bar{0} \end{pmatrix}$

$\updownarrow$

$\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}$ Not a cayley table

Fact: any automorphism must send G to itself

$G = (\mathbb{Z}/3\mathbb{Z}, +) = \{1, a, b\}$

$\text{Aut}(G) = \{\text{identity relabeling},\ a \leftrightarrow b\}$

$\begin{pmatrix} 1 & a & b \\ a & b & 1 \\ b & 1 & a \end{pmatrix} \xleftrightarrow{a \leftrightarrow b\ \checkmark} \begin{pmatrix} 1 & b & a \\ b & a & 1 \\ a & 1 & b \end{pmatrix}$

$G = (\mathbb{Z}/4\mathbb{Z}, +)$ $\begin{pmatrix} 1 & \kappa & \kappa^2 & \kappa^3 \\ \kappa & \kappa^2 & \kappa^3 & 1 \\ \kappa^2 & \kappa^3 & 1 & \kappa \\ \kappa^3 & 1 & \kappa & \kappa^2 \end{pmatrix}$ $\mathrm{Aut}(G) = \{id, \kappa \leftrightarrow \kappa^3\}$

$\kappa \leftrightarrow \kappa^3$ $\quad$ $\kappa \leftrightarrow \kappa^2$ $\quad$ $\kappa^3 \leftrightarrow \kappa^2$

✓ $\quad$ $\begin{pmatrix} 1 & \kappa^2 & \kappa & \kappa^3 \\ \kappa^2 & & \times & \\ \kappa & & & \\ \kappa^3 & & & \end{pmatrix}$ $\quad$ same problem

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Ex: $G = KV_4 = \{1, \leftrightarrow, \updownarrow, \circlearrowleft\}$

$\mathrm{Aut}(G) = \{id, \overset{a}{\updownarrow \to \leftrightarrow}, \overset{b}{\leftrightarrow \to \circlearrowleft}, \overset{c}{\updownarrow \to \circlearrowleft}, l, r\} = \mathrm{Sym}(\Delta)$ ✓

$$\begin{pmatrix} 1 & \leftrightarrow & \updownarrow & \circlearrowleft \\ \leftrightarrow & 1 & \circlearrowleft & \updownarrow \\ \updownarrow & \circlearrowleft & 1 & \leftrightarrow \\ \circlearrowleft & \leftrightarrow & \leftrightarrow & 1 \end{pmatrix} \xrightarrow{\updownarrow \to \leftrightarrow} \begin{pmatrix} 1 & \updownarrow & \leftrightarrow & \circlearrowleft \\ \updownarrow & 1 & \circlearrowleft & \leftrightarrow \\ \leftrightarrow & \circlearrowleft & 1 & \updownarrow \\ \circlearrowleft & \leftrightarrow & \updownarrow & 1 \end{pmatrix}$$