MA453
Homework b

---

1. $R = \mathbb{Z}/3\mathbb{Z}$

   $a = 2$ and $b = 1$

   then $a^2 = 2^2 = 1$ and $b^2 = 1^2 = 1$

2. $(1-a)(1-a) = 1 - a - a + a^2 = a^2 - 2a + 1 = -a + 1 = (1-a)$ because we are given that $a^2 = a$

   so by definition $(1-a)$ is indempotent.

3. Suppose that $x \in \mathbb{Z}/5\mathbb{Z}[\sqrt{3}]$

   then $x = a + b\sqrt{3}$ for some $a, b \in \mathbb{Z}/5\mathbb{Z}$

   so $(a + b\sqrt{3})(a - b\sqrt{3}) = a^2 - ab\sqrt{3} + ab\sqrt{3} - 3b^2 = a^2 - 3b^2$

   so let $x^{-1} = (a - b\sqrt{3})(a^2 - 3b^2)^{-1}$

   then $xx^{-1} = (a + b\sqrt{3})(a - b\sqrt{3})(a^2 - 3b^2)^{-1} = (a^2 - 3b^2)(a^2 - 3b^2)^{-1} = 1$

   so let $c = a(a^2 - 3b^2)^{-1}$ and $d = (-b)(a^2 - 3b^2)^{-1}$

   then $x^{-1} = c + d\sqrt{3}$ and $x^{-1} \in \mathbb{Z}/5\mathbb{Z}[\sqrt{3}]$

   Similarly for $y + k\sqrt{3}$ we can define an inverse in the form of $a + b\sqrt{3} \iff a = y(y^2 - 3k^2)^{-1}$ and $b = -k(y^2 - 3k^2)^{-1}$

4. $\mathbb{Z}/5\mathbb{Z}$ has characteristic of 25 and contains a nonzero element 5 with $5^2 = 0$

   $\mathbb{Z}/5\mathbb{Z}[\sqrt{3}]$ has characteristic 5 and no nonzero element whose square is 0.