MA453
Homework b

1. (a) For any $\vec{x} = (x_1, \ldots, x_m)^T \in \mathbb{Z}^m$

   $\vec{x} = \sum_{i=1}^{m} x_i e_i \Rightarrow \phi(\vec{x}) = \sum_{i=1}^{m} x_i \phi(e_i) = \sum_{i=1}^{m} x_i w_i = W\vec{x}$, where,

   $W = \left[\mathbf{w}_1 \, \mathbf{w}_2 \, \cdots \, \mathbf{w}_m \right]$

   (b) so, $w_1, ..., w_m \in \mathbb{Q} \Rightarrow$ linearly dependent over $\mathbb{Q}$

   so there are rational coefficients $a_1, ..., a_m$ not all 0 with $\sum_{i=1}^{m} a_i w_i = 0$

   (c) so choose $0 \neq b \in \mathbb{Z}$ so that $z = ba \in \mathbb{Z}^m - 0$

   then $Wz = bWa = 0$

   (d) by (a) $\phi(z) = Wz = 0$ with $z \neq 0$

   which contradicts injectivity of ϕ

   so no isomorphism can exist with $m > n$, so $m \leq n$, same idea to isomorphism: $\mathbb{Z}^n \mapsto \mathbb{Z}^m$ gives $n \leq m$

   thus $m = n$

2. $720 = 2^4 \cdot 3^2 \cdot 5$

   $\alpha(720) = part(4) \cdot part(2) \cdot part(1) = 5 \cdot 2 \cdot 1 = 10$

3. (a) part(1) = 1, part(2) = 2, part(3) = 3, part(4) = 5, part(5) = 7, part(6) = 11

(b) An abelian group made of factors of p means every element has order that is some power of p, so only possible ways to make a product of cyclic pieces whose total exponent is e are:
$e = 1 \rightarrow$ splits $p^1 : p \rightarrow$ 1
$e = 2 \rightarrow$ splits $p^2 : p \times p \rightarrow$ 2
$e = 3 \rightarrow$ splits $p^3 : p^2 \times p, p \times p \times p \rightarrow$ 3

which are the exact partitions of part(e), which are the number of ways to break e into positive integer that add up to e.

so if n has prime decomposition: $p^{e_1} \times ... \times p^{e_r}$ then a group of order n must be built by picking:

one choice of structure for $e_1, e_2, ..., e_r$, which do not interfere with each other because the multiplication factors are independent

So the total number of possibilities equals product of the choices for each prime:

$\alpha(n) = part(e_1) \times ... \times part(e_r)$

(c) max $part(e_2)part(e_3)part(e_5)$ subject to $e_2 + e_3 + e_5 = 8$ and $e_i \geq 1$

(6,1,1): part(6)part(1)part(1) = $11 \cdot 1 \cdot 1$ = 11
(5,2,1): $7 \cdot 2 \cdot 1$ = 14
(4,3,1): $5 \cdot 3 \cdot 1$ = 15
(4,2,2): $5 \cdot 2 \cdot 2$ = 20
(3,3,2): $3 \cdot 3 \cdot 2$ = 18

so max $(e_2, e_3, e_5) = (4, 2, 2)$

$n = 2^4 \cdot 3^2 \cdot 5^2 = 16 \cdot 9 \cdot 25 = 3600$

and max number of abelian groups is part(4)part(2)part(2) = 20