MA 453
Homework b

1. Suppose that H is a subgroup of index 2 in a group G.

   so G has exactly two distinct cosets

   H and gH for some $g \notin H$

   so the left coset gH and right coset Hg must be one of each of the two distinct cosets

   so only possibilities are $gH = H$ or $gH = gH$, and $Hg = H$ or $Hg = gH$.

   $gH = Hg$ must hold because the index 2 $\Rightarrow$ coset structure is symmetric

   so for all $h \in H$, $g^{-1}hg \in H \Rightarrow H$ is normal.

2. Case 1: $g = i$

   $g^{-1} = -i$ (since $i \cdot (-i) = -i^2 = -(-1) = 1$). For $h = 1$:

   $g^{-1}hg = (-i) \cdot 1 \cdot i = (-i) \cdot i = -i^2 = -(-1) = 1 \in H$.

   For $h = -1$:

   $g^{-1}hg = (-i) \cdot (-1) \cdot i = i \cdot i = i^2 = -1 \in H$.

   Case 2: $g = -i$

   $g^{-1} = i$ (since $(-i) \cdot i = -i \cdot i = -i^2 = -(-1) = 1$). For $h = 1$:

   $g^{-1}hg = i \cdot 1 \cdot (-i) = i \cdot (-i) = -i^2 = -(-1) = 1 \in H$.

   For $h = -1$:

   $g^{-1}hg = i \cdot (-1) \cdot (-i) = -i \cdot (-i) = i^2 = -1 \in H$.

   Case 3: $g = j$

   $g^{-1} = -j$ (since $j \cdot (-j) = -j^2 = -(-1) = 1$). For $h = 1$:

   $g^{-1}hg = (-j) \cdot 1 \cdot j = (-j) \cdot j = -j^2 = -(-1) = 1 \in H$.

   For $h = -1$:

   $g^{-1}hg = (-j) \cdot (-1) \cdot j = j \cdot j = j^2 = -1 \in H$.

   Case 4: $g = -j$

   $g^{-1} = j$ (since $(-j) \cdot j = -j \cdot j = -j^2 = -(-1) = 1$). For $h = 1$:

   $g^{-1}hg = j \cdot 1 \cdot (-j) = j \cdot (-j) = -j^2 = -(-1) = 1 \in H$.

   For $h = -1$:

   $g^{-1}hg = j \cdot (-1) \cdot (-j) = -j \cdot (-j) = j^2 = -1 \in H$.

   Case 5: $g = k$

   $g^{-1} = -k$ (since $k \cdot (-k) = -k^2 = -(-1) = 1$). For $h = 1$:

   $g^{-1}hg = (-k) \cdot 1 \cdot k = (-k) \cdot k = -k^2 = -(-1) = 1 \in H$.

   For $h = -1$:

   $g^{-1}hg = (-k) \cdot (-1) \cdot k = k \cdot k = k^2 = -1 \in H$.

   Case 6: $g = -k$

   $g^{-1} = k$ (since $(-k) \cdot k = -k \cdot k = -k^2 = -(-1) = 1$). For $h = 1$:

   $g^{-1}hg = k \cdot 1 \cdot (-k) = k \cdot (-k) = -k^2 = -(-1) = 1 \in H$.

   For $h = -1$:

   $g^{-1}hg = k \cdot (-1) \cdot (-k) = -k \cdot (-k) = k^2 = -1 \in H$.

   Case 7: $g = -1$

   $g^{-1} = -1$ (since $(-1) \cdot (-1) = 1$). For $h = 1$:

   $g^{-1}hg = (-1) \cdot 1 \cdot (-1) = (-1) \cdot (-1) = 1 \in H$.

   For $h = -1$:

$g^{-1}hg = (-1) \cdot (-1) \cdot (-1) = 1 \cdot (-1) = -1 \in H.$
so for all $g \in G$ and $h \in H$, $g^{-1}hg \in H$

so $H = 1, -1$ is normal in G

3. so if $H = \{e, a\}$ is normal in S_5 then for all $g \in S_5$, $g^{-1}ag = a \Rightarrow ga = ag$

   by assignment 5b problem 3 we know that a single cycle a does not commute with all $g \in S_5$

   so a cannot be a single cycle.

   consider $a =$(1 2)(3 4)

   (1 2)(3 4)(1 5) = (1 5)(1 2)(3 4) but (1 5)(1 2)(3 4) = (1 5)(1 2)(3 4 5) which is different

   next consider a = (1 2)(3 4 5)

   (1 2)(3 4 5)(1 5) = (1 5)(1 2) (3 4 5) but (1 5)(1 2)(3 4 5) = (1 5)(1 2)(3 4 5) which again differs

   so a must commute with all $g \in S_5$ but these a do not, and thus no such H exists.