MA 453
Homework b

1. Suppose $S = \{g \in G : \varphi(g) = g\}$

   (a) $\varphi(e) = e \in S$

   (b) Suppose that $x, y \in S$

   then $\varphi(x) = x$ and $\varphi(y) = y$

   so, $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) = x \cdot y \in S$

   (c) $\varphi(x^{-1}) = (\varphi(x))^{-1} = (x)^{-1} = x^{-1} \in S$

2. Suppose that $|G| = 91$ and that $\text{ord}(a) \neq \text{ord}(b)$, and that $a, b \neq e_G$

   so by Langrange's thm $\Rightarrow$ $|a|, |b| \in \{7, 13, 91\}$ all but 1 because they can not be the identity.

   so if any of $|a|$ or $|b| = 91$, then we are done because $\langle a, b \rangle = G$

   more interestingly, if $|a| = 7$ and $|b| = 13$ (the other way around works too!)

   let $H = \langle a, b \rangle$

   so if $a \in H$ and $|a| = 7$ then 7 divides $|H|$ and similarly, 13 divides $|H|$

   and since $|H|$ must divide $|G| = 91$ then the only possibility is $|H| = 91$

   $\Rightarrow G = H$

3. Define Φ : left cosets $gH \mapsto$ vertices of cube

   $\Phi(gH) = g(v)$

   Now suppose that $gH = g'H$, then $g' = gh$ for some $h \in H$

   so, h fixes v, and $g'(v) = gh(v) = g(v)$

   so $\Phi(gH) = \Phi(g'H)$, so it is well defined.

   so if $\Phi(gH) = \Phi(g'H)$, then $g(v) = g'(v)$

   so, $(g^{-1}g')(v) = v$

   so, $g^{-1}g' \in H \Rightarrow \Phi$ is injective.

   Next, let ω be any vertex, then by the symmetry of the cube pick a $g \in G$ such that $g(v) = \omega$

   then $\Phi(gH) = \omega$

   so Φ mnust be surjective

   so it is a bijection between the set of left cosets and the set of vertices

   so, $(G : H) = \#\{\text{left cosets}\} = \#\{\text{vertices}\} = 8$

4. Order $3 : \langle r \rangle = \{e, r, r^2\}$

   Order $2 : \langle s \rangle = \{e, s\}, \langle rs \rangle = \{e, rs\}, \langle r^2, s \rangle = \{e, r^2 s\}$

   The identity is normal

   $\langle r \rangle$ is any index 2, so any index 2 subgroup is normal, $srs = r^{-1} \in \langle r \rangle$ and $r \langle r \rangle r^{-1} = \langle r \rangle$

   so, $rsr^{-1} = rsr^2$, using $sr = r^{-1}s$, we get that $rs = sr^{-1} = sr^2$

   so, $rsr^2 = (sr^2)r^2 = sr \neq s$

   so, $r \langle s \rangle r^{-1} = \langle rs \rangle \neq \langle s \rangle$

   so $\langle s \rangle$ is not normal, and similarly for the rest of the two order subgroups.

   Next, the only nontrivial proper normal subgroup is $\langle r \rangle$. There is no normal subgroup of order 2.

   So any decomposition of $D_3 = H_1 \times H_2$ with non trivial factors would need $|H_1| = 3$ and $|H_2| = 2$

   but there is no group such that $|H_2| = 2$

   so there is no such product inside D_3

5. $\phi(7 + 50\mathbb{Z}) = 6 + 15\mathbb{Z}$

   let $u = \phi(1 + 50\mathbb{Z}) \in \mathbb{Z}/15\mathbb{Z}$

   then $7u = \phi(7 \cdot (1 + 50\mathbb{Z})) = \phi(7 + 50\mathbb{Z}) = 6 + 15\mathbb{Z}$

   so, $7u \equiv 6 (\text{ mod } 15)$

   $u \equiv 13 \cdot 6 = 78 \equiv 78 - 75 = 3 (\text{ mod } 15)$

   so $u = 3 + 15\mathbb{Z}$

   so, $\phi(x + 50\mathbb{Z}) = 3x + 15\mathbb{Z}$, for all $x \in \mathbb{Z}$

   Next, $\ker\phi = \{x + 50\mathbb{Z} : \phi(x + 50\mathbb{Z}) = 0 + 15\mathbb{Z}\}$

   so, want $x \equiv 1 (\text{ mod } 5)$

   so $\ker\phi = \{0, 5, 10, 15, 20, 25, 30, 35, 40, 45\} + 50\mathbb{Z}$

   we need $3x \equiv 3 (\text{ mod } 15)$

   so, $3(x - 1) \equiv 0 (\text{ mod } 15)$

   so $5 | (x - 1)$

   so, $x \equiv 1 (\text{ mod } 5)$

   so there are ten classes: $1, 6, 11, 16, 21, 26, 31, 36, 41, 46 (\text{mod } 50)$

   and each satisfies $\phi(x + 50\mathbb{Z}) = 3 + 15\mathbb{Z}$