MA 453
Homework b

1. $5x^4 + 3x^3 + 4 = (4x^2 + 3x)(3x^2 + 2x + 2) = (x + 4)$

   $3x^2 + 2x + 2 = (3x + 4)(x + 4) + 0$

   $gcd(f, g) = x + 4$

2. a) $\mathbb{Z}[x]$

$x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1)$ which are irreducible

$x^4 + 1$ has no linear roots because it has no solution in $\mathbb{Z}$

so, $x^4+1 = (x^2+ax+b)(x^2+cx+d) = x^4+(a+c)x^3+(ac+b+d)x^2+(ad+bc)x+bd$

so our only choice is if $b, d = \pm 1$

but if that happen then we need a $c \in \mathbb{Z}$ such that $c^2 = 2$ which can't happen ouch.

Hence, $x^4 + 1$ is already irreducible

b) $\mathbb{Z}/5\mathbb{Z}$

$x^4 + 1$ has no linear factors and thus it must factor into two quadratics

$x^4 + 1 = (x^2 + ax + b)(x^2 + cs + d)$

$= x^4 + (a + c)x^3 + (ac + b + d)x^2 + (ad + bc)x + bd$

this happens $\iff$ $b, d = 1, 1$ or $2, 3$ or $3, 2$(← I'll omit this case it is the same as 2,3)

so if $b, d = 1 \Rightarrow c^2 = 2$ but there is no such $c \in \mathbb{Z}/5\mathbb{Z}$, ouch

in the case where $b, d = 2, 3$

$c^2 = 5 \iff c = 5$ and $a = 0, b = 2, d = 3$

so, $x^4 + 1 = (x^2 + 2)(x^2 + 5x + 3) = (x^2 + 2)(x^2 + 3)$

$x^4 - 1$ has 4 linear factors

so $x^4 - 1 = (x - 1)(x - 2)(x - 3)(x - 4)$

3. So there are 8 possible choices

$$\{x^2, x^2+x+1, x^2+2x+2, x^2+x+2, x^2+2x+1, x^2+x, x^2+2x, x^2+1, x^2+2\}$$

and out of the possible choices only $\{x^2+2x+2, x^2+x+2, x^2+1\}$ have no roots and thus irreducible monic polynomials.

4. So there are 8 possible choices

$$\{x^3, x^3+x^2+x+1, x^3+x^2+x, x^3+x^2, x^3+x, x^3+1, x^3+x+1, x^3+x^2+1\}$$

and out of the possible choices only $\{x^3+x+1, x^3+x^2+1\}$ are irreducible monic polynomials.