MA 453
Homework b

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1. $\delta((a+bi)(c+di)) = \delta(ac+adi+bci-bd) = \delta((ac-bd)+(ad+bc)i)$

   $= (ac-bd)^2 + (ad+bc)^2 = a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 = a^2(c^2+d^2) + b^2(c^2+d^2)$

   $= (a^2+b^2)(c^2+d^2) = \delta(a+bi)\delta(c+di)$

2. (a) $\dfrac{x}{y} = \dfrac{(a+bi)(c+di)}{c^2-d^2} = \dfrac{(ac+bd)+(bc-ad)i}{c^2+d^2}$

Define $s = \dfrac{ac+bd}{c^2+d^2}$ and $t = \dfrac{bc-ad}{c^2+d^2}$ both $\in \mathbb{Q}$

so $\dfrac{x}{y} = s + ti \in \mathbb{Q}[i]$

(b) choose $m, n \in \mathbb{Z}$ such that $|s-m| \leq \frac{1}{2}$ and $|t-n| \leq \frac{1}{2}$ and let $q = m + ni \in \mathbb{Z}[i]$

$\dfrac{x}{y} = s + ti \Rightarrow x = (s+ti)y = (m+ni)y + ((s-m)+(t-n)i)y$

let $r = ((s-m)+(t-n)i)y$, so $r = x - (m+ni)y = x - qy$

so, $x, y, q \in \mathbb{Z}[i] \Rightarrow r \in \mathbb{Z}[i]$

(c) $\delta(r) = \delta((s-m)+(t-n)i)\delta(y)$

$= ((s-m)^2 + (t-n)^2)\delta(y)$

so, $(s-m)^2 \leq \frac{1}{4}$ and $(t-n)^2 \leq \frac{1}{4}$

$\Rightarrow (s-m)^2 + (t-n)^2 \leq \frac{1}{2}$

$\Rightarrow \delta(r) \leq \frac{1}{2}\delta(y) \leq \delta(y)$

(d) if $r = 0$ then $r \in \mathbb{Z}[i]$ and $0 = \delta(r) < \delta(y) \neq 0$

3. If there is some $a \in \mathbb{Z}_{11}$ such that $f(a) \equiv 0 \bmod 11 \Rightarrow f$ is reducible

   however $a^2 + a + 4 \equiv 0 \bmod 11$ has no solution for $a \in \mathbb{Z}_{11}$

   so it must be irreducible in $\mathbb{Z}_{11}$

4. $5x^4 + 3x^3 + 4 = (\frac{5x^2}{3} - \frac{x}{9} - \frac{28}{27})(3x^2 + 2x + 2) + (\frac{62x}{27} + \frac{164}{27})$

$3x^2 + 2x + 2 = (\frac{81x}{62} - \frac{2484}{961})(\frac{62x}{27} + \frac{164}{27}) + \frac{17010}{961}$

$\frac{62x}{27} + \frac{164}{27} = (\frac{1701}{961})(\frac{29791x}{229635} + \frac{78802}{229635}) + 0$

$gcd(f, g) = \dfrac{17010}{961}$