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Homework a

1. Base case $j = 0, x^i \cdot x^0 = x^{i+0}$

   $x^i = x^i$

   Suppose we've have checked up to $j = k$ case then we know, $x^i \cdot x^k = x^{i+k} \quad (\star)$

   So, $x^i \cdot x^{k+1} = x^i(x^k \cdot x) = (x^i \cdot x^k)x = x^{i+k} \cdot x$ by $(\star)$

   Hence, true for the $j = k + 1$ case

   So by symmetry $x^j \cdot x^i = x^{j+i}$, since $i + j = j + i$

   So, $\underbrace{x \cdots x}_{i \text{ copies}} \underbrace{x \cdots x}_{j \text{ copies}} = x^i \cdot x^j = x^{i+j} = x^{j+i} = x^j \cdot x^i = \underbrace{x \cdots x}_{j \text{ copies}} \underbrace{x \cdots x}_{i \text{ copies}}$

   Next, suppose C_n is the nth cyclic group

   So, $a^i, a^j \in C_n$, where $0 \leq i, j \leq n$

   $a^i \cdot a^j = a^{i+j} = a^{j+i} = a^j \cdot a^i$

   So, C_n is abelian by definition.

2. a) $\text{ord}(a) = \dfrac{n}{gcd(a,n)}$, so the possible order must be the divisors of n

divisors of $364 = \{1, 2, 4, 7, 13, 14, 26, 28, 52, 91, 182, 364\}$

b) ord(0) = 1, ord(182) = 2, ord(4) = 91, ord(7) = 52, ord(13) = 28, ord(14) = 26, ord(26) = 14, ord(28) = 13, ord(52) = 7, ord(91) = 4, ord(182) = 2, ord(364) = 1.

c) order 1 (generator [364]): [0]

order 2 (generator [182]): [0], [182]

order 4 (generator [91]): [0], [91], [182], [273]

order 7 (generator [52]): [0], [52], [104], [156], [208], [260], [312]

order 13 (generator [28]): [0], [28], [56], [84], [112], [140], [168], [196], [224], [252], [280], [308], [336]

order 14 (generator [26]): [0], [26], [52], [78], [104], [130], [156], [182], [208], [234], [260], [286], [312], [338]

order 26 (generator [14]): [0], [14], [28], [42], [56], [70], [84], [98], [112], [126], [140], [154], [168], [182], [196], [210], [224], [238], [252], [266], [280], [294], [308], [322], [336], [350]

order 28 (generator [13]): [0], [13], [26], [39], [52], [65], [78], [91], [104], [117], [130], [143], [156], [169], [182], [195], [208], [221], [234], [247], [260], [273], [286], [299], [312], [325], [338], [351]

order 52 (generator [7]): [0], [7], [14], [21], [28], [35], [42], [49], [56], [63], [70], [77], [84], [91], [98], [105], [112], [119], [126], [133], [140], [147], [154], [161], [168], [175], [182], [189], [196], [203], [210], [217], [224], [231], [238], [245], [252], [259], [266], [273], [280], [287], [294], [301], [308], [315], [322], [329], [336], [343], [350], [357]

order 91 (generator [4]): [0], [4], [8], [12], [16], [20], [24], [28], [32], [36], [40], [44], [48], [52], [56], [60], [64], [68], [72], [76], [80], [84], [88], [92], [96], [100], [104], [108], [112], [116], [120], [124], [128], [132], [136], [140], [144], [148], [152], [156], [160], [164], [168], [172], [176], [180], [184], [188], [192], [196], [200], [204], [208], [212], [216], [220], [224], [228], [232], [236], [240], [244], [248], [252], [256], [260], [264], [268], [272], [276], [280], [284], [288], [292], [296], [300], [304], [308], [312], [316], [320], [324], [328], [332], [336], [340], [344], [348], [352], [356], [360]

order 182 (generator [2]): [0], [2], [4], [6], [8], [10], [12], [14], [16], [18], [20], [22],

[24], [26], [28], [30], [32], [34], [36], [38], [40], [42], [44], [46], [48], [50], [52], [54], [56], [58], [60], [62], [64], [66], [68], [70], [72], [74], [76], [78], [80], [82], [84], [86], [88], [90], [92], [94], [96], [98], [100], [102], [104], [106], [108], [110], [112], [114], [116], [118], [120], [122], [124], [126], [128], [130], [132], [134], [136], [138], [140], [142], [144], [146], [148], [150], [152], [154], [156], [158], [160], [162], [164], [166], [168], [170], [172], [174], [176], [178], [180], [182], [184], [186], [188], [190], [192], [194], [196], [198], [200], [202], [204], [206], [208], [210], [212], [214], [216], [218], [220], [222], [224], [226], [228], [230], [232], [234], [236], [238], [240], [242], [244], [246], [248], [250], [252], [254], [256], [258], [260], [262], [264], [266], [268], [270], [272], [274], [276], [278], [280], [282], [284], [286], [288], [290], [292], [294], [296], [298], [300], [302], [304], [306], [308], [310], [312], [314], [316], [318], [320], [322], [324], [326], [328], [330], [332], [334], [336], [338], [340], [342], [344], [346], [348], [350], [352], [354], [356], [358], [360], [362]

order 364 (generator [1]): [0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], [52], [53], [54], [55], [56], [57], [58], [59], [60], [61], [62], [63], [64], [65], [66], [67], [68], [69], [70], [71], [72], [73], [74], [75], [76], [77], [78], [79], [80], [81], [82], [83], [84], [85], [86], [87], [88], [89], [90], [91], [92], [93], [94], [95], [96], [97], [98], [99], [100], [101], [102], [103], [104], [105], [106], [107], [108], [109], [110], [111], [112], [113], [114], [115], [116], [117], [118], [119], [120], [121], [122], [123], [124], [125], [126], [127], [128], [129], [130], [131], [132], [133], [134], [135], [136], [137], [138], [139], [140], [141], [142], [143], [144], [145], [146], [147], [148], [149], [150], [151], [152], [153], [154], [155], [156], [157], [158], [159], [160], [161], [162], [163], [164], [165], [166], [167], [168], [169], [170], [171], [172], [173], [174], [175], [176], [177], [178], [179], [180], [181], [182], [183], [184], [185], [186], [187], [188], [189], [190], [191], [192], [193], [194], [195], [196], [197], [198], [199], [200], [201], [202], [203], [204], [205], [206], [207], [208], [209], [210], [211], [212], [213], [214], [215], [216], [217], [218], [219], [220], [221], [222], [223], [224], [225], [226], [227], [228], [229], [230], [231], [232], [233], [234], [235], [236], [237], [238], [239], [240], [241], [242], [243], [244], [245], [246], [247], [248], [249], [250], [251], [252], [253], [254], [255], [256], [257], [258], [259], [260], [261], [262], [263], [264], [265], [266], [267], [268], [269], [270], [271], [272], [273], [274], [275], [276], [277], [278], [279], [280], [281], [282], [283], [284], [285], [286], [287], [288], [289], [290], [291], [292], [293], [294], [295], [296], [297], [298], [299], [300], [301], [302], [303], [304], [305], [306], [307], [308], [309], [310], [311], [312], [313], [314], [315], [316], [317], [318], [319], [320], [321], [322], [323], [324], [325], [326], [327], [328], [329], [330], [331], [332], [333], [334], [335], [336], [337], [338], [339], [340], [341], [342], [343], [344], [345], [346], [347], [348], [349], [350], [351], [352], [353], [354], [355], [356], [357], [358], [359], [360], [361], [362], [363]

Number of subgroups 12, I know you probably did not ask for this but it was easy to code, [https://luiscastelds.github.io/subgroups/]

3. Suppose that G is any group and that $g \in G$

   And suppose that, $\text{ord}(g) = n$ and $m = \text{ord}(g^{-1})$

   Since, $g^n = e$

   Then by taking the inverse, $(g^{-1})^n = (g^n)^{-1} = e \Rightarrow m|n$

   Since, $(g^{-1})^m = e$

   Then by takin the inverse, $g^m = e \Rightarrow n|m$

   So, $n|m$ and $m|n \Rightarrow m = n$

4. In the dihedral group D_{21}, the composition of two distinct reflection $R \circ R'$ results in a rotation.

   So the order of the composition depends on the angle between the reflection lines.

   Since D_{21}, has 21 rotational symmetries the possible orders of the rotation are the divisors of 21, which are $\{1, 3, 7, 21\}$.

   This is because the order of a rotation is the smallest positive integer n such that n times the rotation angle is a multiple 2π and for D_{21}, which corresponds to the structure of the group.