MA 453
Homework a

1. Choose a σ that sends v to a different vertex

   Then $\sigma v = w$ for some vertex $w \neq v$

   Now suppose that $h \in H$ and that $h \neq \text{ id}$

   So, $(\sigma h \sigma^{-1})v = \sigma h(\sigma^{-1} v)$

   Case 1: $\sigma^{-1} v = w$

   but $\sigma v = w$, so it contradicts our assumption!

   so $\sigma^{-1} v = v$ can't happen, ouch!

   Case 2: $\sigma^{-1} v \neq v$

   let $\sigma^{-1} v = u \neq v$

   then $\sigma h u \neq v$ because h only fixes v, ouch again!

   so, $\sigma h \sigma^{-1} \in \sigma H \sigma^{-1}$ but $\notin H$

2. $G = \{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\}$

so, $\text{ord}(a) = 3$ because $a^3 = a$

and similarly, $\text{ord}(b) = 2$ because $b^2 = b$

so, $\langle a\rangle = \{\text{id}, a, a^2\}$ and $\langle b\rangle = \{\text{id}, b\}$

so, $aba^{-1} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \notin \langle b\rangle$

so, $\langle b\rangle$ is not normal

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} a \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}^{-1} = a \in \langle a\rangle$$

$$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} a \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}^{-1} = \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix} = a^2 \in \langle a\rangle$$

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$aaa^{-1} = a \in \langle a\rangle$

$$\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix} a \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}^{-1} = \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix} = a^2 \in \langle a\rangle$$

so $\langle a\rangle$ is normal

3. (a) Suppose that every non-identity element of G has order 11

so pick $a \in G$ with $\text{ord}(a) = 11$

Let $H = \langle a \rangle$

then $|H| = 11$

so, $|G/H| = \frac{33}{11} = 3$

since 3 is prime, G/H is cyclic

then there is a $b \in G$ such that bH generates G/H

so $(bH)^3 = H \Rightarrow b^3 \in H$

so assume $\text{ord}(b) = 11$

so $b^3 \in H = \langle a \rangle$

so $b^3 = a^k$ for some k

so $(b^3)^4 = b^{12} = a^{4k} = e$

and since $b^{11} = e$

then $b = a^t$ for some t, so $b \in H$

but $bH = H$ which cannot generate G/H

which is a contradiction, ouch!, so not all non-identity can have order 11.

(b) By the same argument, if every non-identity element had order 3

take a with $\text{ord}(a) = 3$, then $H = \langle a \rangle$ has order 3

and $|G/H| = 11$

then G/H cyclic $\Rightarrow$ there is a b with $b^{11} \in H$

so if b also had order 3, then $(b^{11})^3 = e \Rightarrow b \in H$, contradiction, ouch!

so, not all elements can have order 3

(c) Suppose that $a, b \in G$ with $\text{ord}(a) = 3$ and $\text{ord}(b) = 11$

since G is abelian, $ab = ba$

so, $(ab)^{33} = a^{33}b^{33} = e$

so, $\text{ord}(33)$ divides 33

It can't be 1, because $ab = e$, so $a = b^{-1}$

which is not possible since order 3 and 11 differ

It can't be 3 because $(ab)^3 = e \Rightarrow a^3b^3 = b^3 = e \Rightarrow b = e$ contradiction, ouch!

It can't be 11 because $(ab)^{11} = e \Rightarrow a^{11}b^{11} = a^{11} = e$, but $\text{ord}(a) = 3 \Rightarrow 3|11$ Big ouch!

So, $\text{ord}(ab) = 33$