MA 453
Homework a

1. Let $\psi : \mathbb{Z}/12\mathbb{Z} \to \mathbb{Z}/18\mathbb{Z}$

   $12x = 0$ in $\mathbb{Z}/18\mathbb{Z} \iff 18|12x$

   since $gcd(12, 18) = 6 \Rightarrow 3|2x \Rightarrow x \equiv 0(\mod)3$

   so the possibilities are multiples of 3 in $\mathbb{Z}/18\mathbb{Z}$

   $x \in \{0, 3, 6, 9, 12, 15\}$

   Next, suppose that $\psi(9 + 12\mathbb{Z}) = 9 + 18\mathbb{Z}$

   so $9x = \psi(9 \cdot (1 + 12\mathbb{Z})) = \psi(9 + 12\mathbb{Z}) = 9 \in \mathbb{Z}/18\mathbb{Z}$

   so $x = 3k$ with $k \in \{0, 1, 2, 3, 4, 5\}$

   so $9x = 27k \equiv 9k \equiv 9(\text{mod } 18) \Rightarrow k \equiv 1(\text{mod } 2)$

   so, $x \in \{3, 9, 15\} + 18\mathbb{Z}$

2. $U(8) = \text{Aut}(\mathbb{Z}/8\mathbb{Z}, +)$

   so we want $\psi_a \cdot \psi_b = \psi_{ab(mod8)}$

   and, $\psi_a^2 = id \Rightarrow \psi_{a^2} = \psi_1$

   which is the same equivalent to $a^2 \equiv 1(\mod 8)$

   so possible values are $a = \{1, 3, 5, 7\}$

   similarly, for $U(16), U(32)$

   we have $a = \{1, 3, 5, 7, 9, 11, 13, 15\}$ and $a = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31\}$

   Next, so for $k \geq 3$, $U(2^k)$ has four distinct solutions to $x^2 \equiv 1(\mod 2^k)$

   Namely, $x \equiv 1, -1, 1 + 2^{k-1}, -1 + 2^{k-1}$

   so in a cyclic group of order n, the equation $x^2 = 1$ has exactly $gcd(2, n)$ solutions so at most 2 solutions

   since $U(2^k)$ has 4 solutions to $x^2 = 1$ when $k \geq 3$ and hence it cannot be cyclic.

3. $\varphi : \mathbb{Z}/24\mathbb{Z} \to \mathbb{Z}/48\mathbb{Z}$

   $\varphi(a \bmod 24) = 6a \bmod 48$

   $6a \equiv 0 (\bmod 48) \iff 48|6a \iff 8|a$

   so divisors of 8 are $0, 8, 16$

   so ker$\varphi = \{0 \bmod 24, 8 \bmod 24, 16 \bmod 24\}$

   Next, $a \bmod 24 \mapsto 3a \bmod 48$

   Not well defined because, $0 \equiv 24 (\bmod 24)$ which is the same class in $\mathbb{Z}/24\mathbb{Z}$

   but $3 \cdot 0 \equiv 0 (\bmod 48)$ while $3 \cdot 24 = 72 \equiv 24 (\bmod 48)$

   So, the same input class has two different output class, Ouch!

4. D_4 are the symmetries of the square

   Let $R = \{0°, 90°, 180°, 270°\}$

   so, $rr^kr^{-1} = r^k \in R$

   and for reflections,

   $srs^{-1} = r^{-1} \Rightarrow sr^ks^{-1} = (r^{-1})^k = r^{-k} \in R$

   so we always land back in R

   Therefore, the rotation subgroup R is normal in D_4