MA453
Homework a

1. ”⇒” Suppose that $N(\alpha) = 1$

   then $a^2 + 5b^2 = 1 \iff a = \pm 1$ and $b = 0$ for $a, b \in \mathbb{Z}$

   so, $\alpha = a + b\sqrt{-5} = \pm 1$

   ”⇐” Suppose $\alpha = \pm 1 \Rightarrow a + b\sqrt{-5} = 1 \iff a = \pm 1$ and $b = 0$

   so $N(\pm 1 + 0\sqrt{-5}) = (\pm 1)^2 + 5(0^2) = 1$

   So, $N(\pm 1) = 1$ and $a^2 + 5b^2 = 1$ has solution $\pm 1 \Rightarrow$ only units in $\mathbb{Z}[\sqrt{-5}]$

2. Suppose that there is some $\alpha \in R$ for which $N(\alpha) = 2 \Rightarrow$ there are some $a, b \in \mathbb{Z}$ such that $a^2 + b^2\sqrt{-5} = 2 + 0\sqrt{-5}$

   $\Rightarrow a^2 = 2$ and $b^2\sqrt{-5} = 0\sqrt{-5}$

   but there is no $a \in \mathbb{Z}$ such that $a^2 = 2$, so there is no $\alpha \in R$ such that $N(\alpha) = 2$ and thus it can't happen. Big ouch!

   Similarly, there is no $a \in \mathbb{Z}$ such that $a^2 = 3$

   so there is no $\alpha \in R$ such that $N(\alpha) = 3$ and thus it can't happen. Big second ouch!

3. so, $2 = 2 + 0\sqrt{-5}$

   so $N(2) = 2^2 = 4$ so suppose there are $\alpha\beta = 2 \Rightarrow 4 = N(\alpha)N(\beta)$ by week 10b

   so either $N(\alpha), N(\beta) = 1, 4$ or $2, 2$

   by exercise 1 ⇒ 2,2 can't happen.

   Hence one factor has norm 1 (a unit) ⇒ 2 is irreducible.

   3 is the same (trust me bro)

   Next, if $1 \pm \sqrt{-5} = \alpha\beta \Rightarrow N(\alpha)N(\beta) = 6$

   $N(\alpha), N(\beta) = (2, 3), (3, 2), (6, 1), (1, 6)$

   $N(\alpha), N(\beta) = (2, 3), (3, 2)$ can't happen. Big ouch!

   so either (6,1) or (1,6) must happen and thus one factor has norm 1 (unit) ⇒ $1 \pm \sqrt{-5}$ is irreducible

4. We know that $2, 3, 1 \pm \sqrt{-5}$ are irreducible in R

   so, $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$

   so, $N(2) = 4 \neq 6$ and $N(3) = 9 \neq 6$ abd $N(1 \pm \sqrt{-5}) = 6$

   so they are different factorizations!

   so, 6 can be factored into two irreducibles

   hence R is not a UFD