MA 453
Homework a

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1. Thm X.15 $\Rightarrow$ I is generated by one element because it is an ideal ring in $\mathbb{Z}/n\mathbb{Z}$ $\Rightarrow$ I is principal by definition.

2. $x^2 = -x - 1 \equiv x^2 = 2x + 2$ in mod 3

   so we have elements $\{a + bx : a, b \in \{0, 1, 2\}\}$

   so $x^2 + x + 1 = (x - 1)(x - 2) \equiv (x + 2)(x + 1)$

   those our elements are: $(x + 2)(x + 1)$

3. $f : \mathbb{Z}/m\mathbb{Z} \mapsto \mathbb{Z}/n\mathbb{Z}$

$e = f(1 + m\mathbb{Z}) \in \mathbb{Z}/n\mathbb{Z}$

so, $e^2 = e \in \mathbb{Z}/n\mathbb{Z}$

and $m \cdot e = 0 + n\mathbb{Z}$

so $f(m \cdot (1 + m\mathbb{Z})) = 0$

thus, $f(a + m\mathbb{Z}) = a \cdot e + n\mathbb{Z}$ for all $a \in \mathbb{Z}$

4. Suppose that $f : \mathbb{R} \mapsto \mathbb{R}$ and that $f(1) = 1, f(0) = 0$

(1) $0 = f(0) = f(1 + (-1)) = f(1) + f(-1) = 1 + f(-1) \Rightarrow f(-1) = -1$

(2) let $a > 0 \Rightarrow a = x^2$ for some $x \in \mathbb{R}$

$f(a) = f(x^2) = f(x)f(x) = f(x)^2 > 0$

(3) $a < 0 \Rightarrow a = -b$, where $b > 0$

$f(a) = f(-b) = f(-1 \cdot b) = f(-1) \cdot f(b) = -f(b)$

and since $f(b) > 0 \Rightarrow -f(b) < 0$

so, $f(a) < 0$

(4) $a > b \Rightarrow a - b > 0$

$f(a + (-b)) = f(a) + f(-b) > 0$ because $a - b > 0$

$f(-b) > -f(a) \Rightarrow -f(-b) < f(a) \Rightarrow - \cdot (-1)f(b) < f(a) \Rightarrow f(a) > f(b)$

(5) $f(n) = f(\underbrace{1 + ... + 1}_{n \text{ times}}) = \underbrace{1 + ... + 1}_{n \text{ times}} = n$

$f(-n) = f(\underbrace{-1 - ... - 1}_{n \text{ times}}) = \underbrace{-1 - ... - 1}_{n \text{ times}} = -n$

(6) $q = \frac{m}{n}$ with $n \neq 0$ and $m = nq$ and $q \in \mathbb{Q}$

$f(n) \cdot f(q) = f(n \cdot q) = f(m) = n \cdot f(q)$ because $n \in \mathbb{Z}$

$f(m) = m \Rightarrow m = n \cdot f(q) \Rightarrow \frac{m}{n} = f(q) = q$

(7) Suppose that $f(r) \neq r$ for some $r \in \mathbb{R}$

Case 1: $f(r) > r$, then pick $q \in \mathbb{Q}$ such that $r < q < f(r)$ ($\mathbb{Q}$ is dense)

so $q > r \Rightarrow f(q) = q > f(r)$ but ouch this is a contradiction

Case 2: $f(r) < r$, then pick $q \in \mathbb{Q}$ such that $f(r) < q < r$ ($\mathbb{Q}$ is dense)

so $r > q \Rightarrow f(r) = f(q) = q < r$ but ouch this is a contradiction

so $f(r) = r$ for all r