MA453
Homework A

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1. Suppose that $x, y \in G$ and that $x^{34} = y$, and $x^{99} = 1$

   $gcd(34, 99) = 1$

   $99 = 2 \cdot 34 + 31$

   $34 = 1 \cdot 31 + 3 \Rightarrow 1 = 11 \cdot 99 - 32 \cdot 34$

   $31 = 10 \cdot 3 + 1$

   So, $34(-32) + 99(11) = 1$

   So, $x = x^1 = x^{34(-32)+99(11)} = (x^{34})^{-32} = y^{-32} \cdot 1 = y^{-32}$

   We can generalize, if $x^a = y$ and $x^b = 1$

   Let $1 = gcd(a, b)$, choose integers r, s with

   $ra + sb = d$

   So, $x = (x^a)^r (x^b)^s = y^r$

2. Suppose a group G contains an element a with $a^6 = 1$

   So, $a^6 = 1$ and for any k with $gcd(6, k) = 1$

   So, there are integeres r, s such that $6r + ks = 1$

   So, $a^1 = a^{6r+ks} = (a^6)^r \cdot (a^s)^k = (a^s)^k$

   Therefore, $b = a^s$

3. $C_6 = \{1, a, a^2, a^3, a^4, a^5\}$ $\quad$ Aut$(C_6) = \{$Identity, $\phi(a) \mapsto a^5\}$

1) Case: $\phi(a) = 1$

   Then, $\phi(a^n) = \phi(a)^n = 1$

   Ouch! not in Aut(C_6)

2) Case: $\phi(a) = a$

   Identity!

   Trivially in Aut(C_6)

3) Case: $\phi(a) = a^2$

   $\phi(1) = 1, \phi(a) = a^2, \phi(a^2) = a^4, \phi(a^3) = 1, \phi(a^4) = a^2, \phi(a^5) = a^4$

   Ouch! not in Aut(C_6)

4) Case: $\phi(a) = a^3$

   $\phi(1) = 1, \phi(a) = a^3, \phi(a^2) = 1, \phi(a^3) = a^3, \phi(a^4) = 1, \phi(a^5) = a^3$

   Ouch! not in Aut(C_6)

5) Case: $\phi(a) = a^4$

   $\phi(1) = 1, \phi(a) = a^4, \phi(a^2) = a^2, \phi(a^3) = 1, \phi(a^4) = a^4, \phi(a^5) = a^2$

   Ouch! not in Aut(C_6)

6) Case: $\phi(a) = a^5$

   $\phi(1) = 1, \phi(a) = a^5, \phi(a^2) = a^4, \phi(a^3) = a^3, \phi(a^4) = a^2, \phi(a^5) = a$

   Aha! in Aut(C_6)

4. (a) $1 \cdot g = g \cdot 1 = g \Rightarrow 1 \in Z(G)$

Suppose that $x, y \in Z(G)$

then, $xg = gx$, for all $g \in G$

similarly, $yg = gy$, for all $g \in G$

(b) so, $(xy)g = x(yg) = x(gy) = (xg)y = (gx)y = g(xy)$

so, $(xy)g = g(xy) \Rightarrow xy \in Z(G)$

(c) $x^{-1}xg = x^{-1}gx$

$g = x^{-1}gx$

$gx^{-1} = x^{-1}g \Rightarrow x^{-1} \in Z(G)$