(1) $E_1 E_2 E_3 E_4 d = b$ $\qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$E_1 w = b$, $E_2 v = w$, $E_3 u = v$, $E_4 d = u$

$$\begin{cases} w_1 + 3w_2 = 1 \\ 0.5w_2 = 2 \\ 4w_2 + w_3 = 3 \end{cases} \Rightarrow \begin{cases} w_2 = 4 \\ w_3 = 3 - 16 = -13 \\ w_1 = 1 - 12 = -11 \end{cases}$$

$w = (-11, 4, -13)^T$

$$\begin{cases} 2v_1 = -11 \\ v_1 + v_2 = 4 \\ 4v_1 + v_3 = -13 \end{cases} \Rightarrow \begin{cases} v_1 = -11/2 = -5.5 \\ v_2 = 4 - (-5.5) = 9.5 \\ v_3 = -13 - 4(-5.5) = 9 \end{cases}$$

$V = (-5.5, 9.5, 9)^T$

$$\begin{cases} u_1 + u_3 = -5.5 \\ u_2 + 3u_3 = 9.5 \\ u_3 = 9 \end{cases} \Rightarrow \begin{cases} u_3 = 9 \\ u_1 = -5.5 - 9 = -14.5 \\ u_2 = 9.5 - 27 = -17.5 \end{cases}$$

$u = (-14.5, -17.5, 9)^T$

$$\begin{cases} -0.5 d_1 = -14.5 \\ 3d_1 + d_2 = -17.5 \\ d_1 + d_3 = 9 \end{cases} \Rightarrow \begin{cases} d_1 = 29 \\ d_2 = -17.5 - 3 \cdot 29 = -104.5 \\ d_3 = 9 - 29 = -20 \end{cases}$$

$$d = \begin{pmatrix} 29 \\ -104.5 \\ -20 \end{pmatrix}$$

$yE = r$, $E = E_1 E_2 E_3 E_4$ and $r = (1,2,3)$

$(yE)^T = E^T y^T = r^T$

$E_4^T z_1 = r^T$, $E_3^T z_2 = z_1$, $E_2^T z_3 = z_2$, $E_1^T y^T = z_3$

$$\begin{cases} -0.5a + 3b + c = 1 \\ b = 2 \\ c = 3 \end{cases} \Rightarrow a = 16,\ b = 2,\ c = 3 \quad z_1 = (16, 2, 3)^T$$

$$\begin{cases} p = 16 \\ q = 2 \\ p + 3q + r = 3 \end{cases} \Rightarrow r = 3 - 16 - 6 = .19, \quad z_2 = (16, 2, -19)^T$$

$$\begin{cases} 2u + v + 4w = 16 \\ v = 2 \\ w = -19 \end{cases} \Rightarrow z_3 = (45, 2, -19)^T$$

$$\begin{cases} y_1 = 45 \\ 3y_1 + 0.5y_2 + 4y_3 \\ y_3 = -19 \end{cases} \Rightarrow y = (45, -114, -19)$$

(2)

$$\max \quad 3x_1 + 4x_2 = 12$$
$$-x_1 + 2x_2 \leq 2$$
$$x_1 + 4x_2 \geq 6$$
$$x_1, x_2 \geq 0$$

$$\max \ z = -2x_1 - x_2 - My_1 - My_2$$

$$3x_1 + 4x_2 + y_1 = 12$$
$$-x_1 + 2x_2 + w_1 = 2$$
$$x_1 + 4x_2 - w_2 + y_2 = 6$$
$$x_1, x_2, w_1, w_2, y_1, y_2 \geq 0$$

$$y_1 = 12 - 3x_1 - 4x_2$$
$$w_1 = 2 + x_1 - 2x_2 \qquad z = -2x_1 - x_2 - My_1 - My_2$$
$$y_2 = 6 - x_1 - 4x_2 + w_2 \qquad z = -18M + (4M-2)x_1 + (8M-1)x_2 - Mw_2$$

enter x_2, leave w_1:

$y_1 = 12 - 4x_2 \Rightarrow x_2 \leq 3$, $w_1 = 2 - 2x_2 \Rightarrow x_2 \leq 1$, $y_2 = 6 - 4x_2 \Rightarrow x_2 \leq 1.5$

$$x_2 = 1 + \frac{1}{2}x_1 - \frac{1}{2}w_1$$

enter x_1, leave y_2

$y_1 = 8 - 5x_1 \Rightarrow x_1 \leq \frac{8}{5}$, $x_2 = 1 + \frac{1}{2}x_1$, $y_2 = 2 - 3x_1 \Rightarrow x_1 \leq \frac{2}{3}$

$y_2 = 2 - 3x_1 + 2w_1 + w_2$, $\quad x_1 = \dfrac{2 + 2w_1 + w_2 - y_2}{3}$

$$y_1 = \frac{14}{3} - \frac{4}{3}w_1 - \frac{5}{3}w_2 + \frac{5}{3}y_2$$
$$x_2 = \frac{4}{3} - \frac{1}{6}w_1 + \frac{1}{6}w_2 - \frac{1}{6}y_2 \qquad z = -\frac{14}{3}M - \frac{11}{6} + \left(\frac{4}{3}M - \frac{7}{6}\right)w_1 + \left(\frac{5}{3}M - \frac{5}{6}\right)w_2$$
$$x_1 = \frac{2}{3} + \frac{2}{3}w_1 + \frac{1}{3}w_2 - \frac{1}{3}y_2 \qquad + \left(-\frac{8}{3}M + \frac{5}{6}\right)y_2$$

enter w_2, leave y_1 :

$$y_1 = \frac{14}{3} - \frac{5}{3}w_2 \Rightarrow w_2 \le \frac{14}{5}, \quad x_2 = \frac{4}{3} + \frac{1}{6}w_2, \quad x_1 = \frac{2}{3} + \frac{1}{3}w_2$$

$$y_1 = \frac{14}{3} - \frac{4}{3}w_1 - \frac{5}{3}w_2 + \frac{5}{3}y_2$$

$$w_2 = \frac{14}{5} - \frac{4}{5}w_1 + y_2 - \frac{3}{5}y_1$$

$$x_2 = \frac{9}{5} - \frac{3}{10}w_1 - \frac{1}{10}y_1$$

$$x_1 = \frac{8}{5} + \frac{2}{5}w_1 - \frac{1}{5}y_1$$

$$w_2 = \frac{14}{5} - \frac{4}{5}w_1 + y_2 - \frac{3}{5}y_1$$

$$\zeta = -\frac{25}{6} + \underbrace{\left(-\frac{1}{2}\right)}_{<0}w_1 + \underbrace{\left(-M + \frac{1}{2}\right)}_{<<0}y_1 + \underbrace{(-M)}_{<<0}y_2$$

optimal

$$x_1 = \frac{8}{5}, \quad x_2 = \frac{9}{5}, \quad w_2 = \frac{14}{5}$$

$$\zeta^* = -2\left(\frac{8}{5}\right) - \left(\frac{9}{5}\right) = -5$$

$$x_1^* = \frac{8}{5}, \quad x_2^* = \frac{9}{5}$$

(3)

a) $\min\{ f_1(x) = x,\ f_2(x) = -x \}$ w/ $1 \le x \le 2$

f_1 is minimized by smaller x (push to 1)

$f_2 = -x$ is mimized by larger x (push to 2)

pull on opposite directions on $[1,2]$. any moves that improves one worsens the other

Pareto set: $\{x : 1 \le x \le 2\}$

b) Both objectives have positive coefficients on both variables, so increasing either x_1 or x_2 worsens both objectives. Thus, the common best is at the lower left corner.

Pareto set: $(x_1, x_2) = (1,1)$ only.

c) f_1 minimizes by pushing both x_1, x_2 down

f_2 minimize by pushing x_1 up (coeff. -1) and x_2 down.

so both agree x_2 should be as small as possible ($x_2 = 1$), they conflict on x_1 (one wants 1, the other needs 2).

So any point w/ $x_2 \ge 1$ is dominated by lowering x_2 to 1 while keeping x_1 fixed. Along $x_2 = 1$, moving x_1 creates a trade-off

Pareto set: $\{ (x_1, 1) : 1 \le x_1 \le 2 \}$

d) z_1 smaller x_1 and smaller x_2

z_2 larger x_1, smaller x_2

$z_3 = -x_2$, larger x_2

So conflicts:

x_1: z_1 wants small, z_2 wants large

x_2: z_1, z_2 want small, z_3 wants large

So given the conflicts, no feasible point can be strictly improved in all 3 objectives at once.

Pareto set: $\{(x_1, x_2) : 1 \le x_1 \le 2, 1 \le x_2 \le 2\}$