[V] #5.7 $\max\ 2x_1 - 6x_2$

$-x_1 - x_2 - x_3 \leq -2$ $\qquad x_1, x_2, x_3 \geq 0$

$2x_1 - x_2 + x_3 \leq 1$

add slacks $s_1, s_2 \geq 0$

$-x_1 - x_2 - x_3 + s_1 = 2$

$2x_1 - x_2 + x_3 + s_2 = 1$

	x_1	x_2	x_3	
s_1	-2	1	1	1
s_2	1	2	1	-1
z	0	2	-6	0

primal-infeasible $b = (-2, 1)$

not dual feasible

Phase I:

$A^T y = (-y_1 + 2y_2, -y_1 - y_2, -y_1 + y_2) = (2, -1, 1)$

$c - A^T y = (2, -6, 0) - (2, -1, 1) = (0, -5, -1) \leq 0$

	x_1	x_2	x_3	
s_1	-2	1	1	1
s_2	1	-2	1	-1
z	1	0	-5	-1

$x_1 = 2 - x_2 - x_3 + s_1$

	x_2	x_3	s_1	
x_1	2	-1	-1	1
s_2	-3	3	1	-2
z	1	-5	-1	0

$x_3 = 3 - 3x_2 + 2s_1 + s_2$

	x_2	s_1	s_2	
x_1	-1	2	-1	-1
x_3	1.5	-1.5	0.5	-0.5
z	-2	-2	-2	-1

$x_2 = \frac{1}{2} + \frac{1}{2}x_1 + \frac{1}{2}s_1 + \frac{1}{2}s_2$

	x_1	s_1	s_2	
x_2	0.5	0.5	0.5	
x_3	1.5	-1.5	0.5	-0.5
z	-3	-1	-3	-2

$$x^* = (x_1, x_2, x_3) = (0, 0.5, 1.5), \quad z^* = 2(0) - 6\cdot(0.5) = -3$$

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 2 & -1 & 1 \end{bmatrix}, \quad B = [a_2, a_3] = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}, \quad c_B = (-6, 0)$$

$$B^{-1} = -\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}, \quad y^* = (3, 3)$$

$$A^T y^* = (3, -6, 0) \geq (2, -6, 0) = c$$

$$b^T y^* = (-2, 1)\cdot(3, 3) = -6 + 3 = -3 = z^*$$

$$x_1^* = 0, \quad x_2^* = 0.5, \quad x_3^* = 1.5, \quad z = -3$$

[V] #5.8

$$\max \; -x_1 - 3x_2 - x_3$$

$$2x_1 - 5x_2 + x_3 \le -5 \qquad x_1, x_2, x_3 \ge 0$$

$$2x_1 - x_2 + 2x_3 \le 4$$

$s_1, s_2 \ge 0$;

$$2x_1 - 5x_2 + x_3 + s_1 = -5$$

$$2x_1 - x_2 + 2x_3 + s_2 = 4$$

	x_1	x_2	x_3	
s_1	-5	-2	5	-1
s_2	4	-2	1	-2
z	0	-1	-3	-1

primal infeasible

Phase I: $x_2 = 1 + \frac{2}{5}x_1 + \frac{1}{5}x_3 + \frac{1}{5}s_1$

		x_1	x_2	s_1
x_2	1	$2/5$	$1/5$	$1/5$
s_2	5	$-8/5$	$-9/5$	$1/5$
z	-3	$-11/5$	$-8/5$	$-3/5$

$x_1 = x_3 = s_1 = 0$

$x_2 = 1, \; s_2 = 5$

$x^* = (0,1,0), \; z^* = -3$

$$B = \begin{bmatrix} -5 & 0 \\ -1 & 1 \end{bmatrix}, \; c_B = (-3, 0), \; B^{-1} = \frac{1}{-5}\begin{bmatrix} 1 & 0 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} -1/5 & 0 \\ -1/5 & 1 \end{bmatrix}$$

$$y^{*T} = c_B^T B^{-1} = (-3, 0)\begin{bmatrix} -1/5 & 0 \\ -1/5 & 1 \end{bmatrix} = (3/5, 0)$$

[V] # 5.9 max $x_1 + 3x_2$ $\quad x_1, x_2 \geq 0$

$-x_1 - x_2 \leq -3$

$-x_1 + x_2 \leq -1$ $\quad s_1, s_2, s_3 \geq 0$

$x_1 + 2x_2 \leq 4$

$s_1, s_2, s_3 \geq 0$

	x_1	x_2	
s_1	-3	1	1
s_2	-1	1	-1
s_3	4	-1	-2
ζ	0	1	3

primal infeasible

$y = (0, 0, 3/2)$

$A^T y = (-0 - 0 + 1.5, -0 + 0 + 3) = (1.5, 3)$

$c - A^T y = (1, 3) - (1.5, 3) = (-0.5, 0)$

	x_1	x_2	
s_1	-3	1	1
s_2	-1	1	-1
s_3	4	-1	-2
ζ	6	-0.5	0

$=$

$s_1 = -3 + x_1 + x_2 \Rightarrow x_2 = 3 - x_1 + s_1$

	x_1	s_1	
x_2	3	-1	1
s_2	-4	2	-1
s_3	-2	1	-2
ζ	6	-0.5	0

$S_2 = -4 + 2x_1 - S_1 \Rightarrow x_1 = 2 + \frac{1}{2} s_1 + \frac{1}{2} s_2$

		s_1	s_2
x_1	2	1/2	1/2
x_2	1	1/2	-1/2
s_3	0	-3/2	1/2
z	5	-1/4	-1/4

$x_1^* = 2,\ x_2^* = 1,\ s_3^* = 0$

$z^* = 1 \cdot 2 + 3 \cdot 1 = 5$

$A = \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 1 & 2 \end{bmatrix},\ b = (-3, -1, 4),\ c = (1, 3)$

$\begin{cases} -y_1 - y_2 + y_3 = 1 \\ -y_1 + y_2 + 2y_3 = 3 \end{cases}$

$y^* = (1, 0, 2)$

$b^T y^* = -3(1) - 1(0) + 4(2) = 5 = z^*$

$x_1^* = 2,\ x_2^* = 1,\ z = 5$

[V] #6.1

(a) Indices of Basic Vars: $B = \{3, 1\}$

Basic Vars: $\{X_3, X_1\}$

Indices of Non-Basic Vars: $N = \{4, 2\}$

Non-Basic Vars: $\{X_4, X_1\}$

(b) $X_B^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

(c) $Z_N^* = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

(d) $B^{-1}N = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 1 & -3 \end{bmatrix}$

(e) yes, primal soln. associated w/ dictionary is feasible bc $X_B^* > 0$

(f) NO, the dictionary is not optimal because, $Z_2^* < 0$, ouch

(g) yes, the dictionary is degenerate bc $X_1^* = 0$

[V] #6.2 $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 7 & 5 & -3 & -2 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \end{bmatrix}$

(a) $B = \begin{bmatrix} 3 & 5 \\ -3 & 0 \end{bmatrix}$ (b) $N = \begin{bmatrix} 1 & 2 & 4 \\ 7 & 5 & -2 \end{bmatrix}$

(c) $b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (d) $c_B = \begin{bmatrix} 4 \\ 16 \end{bmatrix}$ (e) $c_N = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}$

(f) $B^{-1}N$, $\det(B) = 3 \cdot 0 - (5)(-3) = 15$

$$B^{-1} = \frac{1}{15}\begin{bmatrix} 0 & -5 \\ 3 & 3 \end{bmatrix}$$

$$B^{-1}N = \frac{1}{15}\begin{bmatrix} 0 & -5 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 7 & 5 & -2 \end{bmatrix} = \begin{bmatrix} -7/3 & -5/3 & 2/3 \\ 8/5 & 7/5 & 2/5 \end{bmatrix}$$

(g) $x_B^* = B^{-1}b = \frac{1}{15}\begin{bmatrix} 0 & -5 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/5 \end{bmatrix}$ $x_3 = 0$

$x_5 = 2/5$

(h) $z^* = c_B^T B^{-1} b = [4 \ 16]\begin{bmatrix} 0 \\ 2/5 \end{bmatrix} = 32/5$

(i) $(B^{-1}N)^T c_B = \begin{bmatrix} -7/3 & 8/5 \\ -5/3 & 7/5 \\ 2/3 & 2/5 \end{bmatrix}\begin{bmatrix} 4 \\ 16 \end{bmatrix} = \begin{bmatrix} 244/15 \\ 236/15 \\ 136/15 \end{bmatrix}$

$$z^* = \begin{bmatrix} 244/15 - 1 \\ 236/15 - 2 \\ 136/15 - 8 \end{bmatrix} = \begin{bmatrix} 229/15 \\ 206/15 \\ 16/15 \end{bmatrix}$$

(j) $x_B = B^{-1}b - B^{-1}N x_N$ & $z = z^* - z_N^{*T} x_N$, $x_N = (x_1, x_2, x_4)^T$

$$x_3 = \frac{7}{3}x_1 + \frac{5}{3}x_2 - \frac{2}{3}x_4$$

$$x_5 = \frac{2}{5} - \frac{8}{5}x_1 - \frac{7}{5}x_2 - \frac{2}{5}x_4$$

$$z = \frac{32}{5} - \left(\frac{229}{15}x_1 + \frac{206}{15}x_2 + \frac{16}{15}x_4\right)$$

[V] #7.1

(a) $\Delta c^T = [2\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]$

$N = \{1\ 5\ 6\ 4\}$ $\quad \Delta c_N^T = [2\ \ 0\ \ 0\ \ 0]$

$B = \{2\ 3\ 7\}$ $\quad \Delta c_B^T = [0\ \ 0\ \ 0]$

$z_N^{*T} = [1.2\ \ 0.2\ \ 0.9\ \ 2.8]$

$\Delta z_N^{*T} = (\underbrace{(B^{-1}N)^T \Delta c_B}_{0} - \Delta c_N^T) = [-2\ \ 0\ \ 0\ \ 0]$

$(z_N^* + \Delta z_N^*)^T = [-0.8\ \ 0.2\ \ 0.9\ \ 2.8]$

$z_1^* + \Delta z_1^* < 0 \Rightarrow$ Not optimal! auch $\frac{1}{2}$

Next, find optimal by simplex method.

$B^{-1}Ne_1 = \begin{bmatrix} 1 \\ 0.2 \\ 1.6 \end{bmatrix}, \quad x_B^{*T} = \begin{bmatrix} 6 \\ 0.4 \\ 11.2 \end{bmatrix}$

$\frac{6}{1} = 6 \Rightarrow i = 2$

$\frac{0.4}{0.2} = 2 \Rightarrow i = 3$

$\frac{11.2}{1.6} = 7.25 \Rightarrow i = 7$

Choose i as leaving var.

so, $N = \{3\ 5\ 6\ 4\}$ $\quad B = \{2\ 1\ 7\}$

So, find new B & N

$$A = \begin{bmatrix} 2 & 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 4 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \quad c^T = [3\ 2\ 1\ 1\ 0\ 0\ 0]$$

$$b = \begin{bmatrix} 8 \\ 12 \\ 18 \end{bmatrix}$$

$$c_B^T = [2\ 3\ 0] \quad c_N^T = [1\ 1\ 0\ 0]$$

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 5 & 1 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}, \quad (B^{-1}N)^T = \begin{bmatrix} -5 & 5 & -8 \\ 3 & -1 & 0 \\ -1 & 1 & -2 \\ 1 & -1/2 & 1/2 \end{bmatrix}$$

$$X_B^* = B^{-1}b = \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix} \quad ((B^{-1}N)^T c_B - c_N)^T = [4\ 2\ 1\ \tfrac{1}{2}]$$

$$z^* = c_B^T B^{-1} b = 14$$

So, we have optimal dict.

$$(x_1, x_2, x_3, x_4) = (2, 4, 0, 0), \quad z^* = 14$$

(b) $\Delta c^T = [0 \quad 0 \quad -0.5 \quad 0 \quad 0 \quad 0 \quad 0]$

$B = \{2 \ 3 \ 7\}$

$$(B^{-1}N)^T = \begin{bmatrix} 1 & 0.2 & 1.6 \\ 0 & 0.2 & -0.4 \\ 0.5 & -0.1 & -0.3 \\ 2 & -0.2 & -1.6 \end{bmatrix}$$

$N = \{1 \ 5 \ 6 \ 4\}$

$\Delta c_B^T = [0 \quad -0.5 \quad 0] \quad \Delta c_N^T = [0 \quad 0 \quad 0 \quad 0]$

$z_N^{*T} = [1.2 \quad 0.2 \quad 0.9 \quad 2.8] \quad \Delta z_N^{*T} = [-0.1 \quad -0.1 \quad 0.05 \quad 0.1]$

$(z_N^* + \Delta z_N^*)^T = [1.1 \quad 0.1 \quad 0.95 \quad 2.9] \geq 0$ dict. is still optimal

New optimal value,

$$(c_B^T + \Delta c_B^T) B^{-1} b = \zeta^* + \Delta c_B^T B^{-1} b$$
$$= 12.4 + (-0.2) = 12.2$$

$$B^{-1} b = \begin{bmatrix} 6 \\ 2/5 \\ 56/5 \end{bmatrix}$$

$(x_1, x_2, x_3, x_4) = (0, 6, 0.4, 0) \quad \zeta^* = 12.2$

(c) $B = \{2 \ 3 \ 7\}$

$N = \{1 \ 5 \ 6 \ 4\}$ $\quad B = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ $\quad B^{-1} = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/5 & -1/10 & 0 \\ -2/5 & -3/10 & 1 \end{bmatrix}$

$x_B^{*T} = [6 \ \ 0.4 \ \ 11.2]$

$\Delta x_B^{*T} = B^{-1}\Delta b = [7 \ \ -1.4 \ \ -4.2]$ $\qquad \Delta b = [0 \ \ 14 \ \ 0]^T$

$$\underbrace{(x_B^* + \Delta x_B^*)^T}_{<0} = [13 \ \ -1 \ \ 7]$$

not optimal

$B = \{2 \ 3 \ 7\}$ $\qquad x_B^{*T} = [13 \ \ -1 \ \ 7]$

$N = \{1 \ 5 \ 6 \ 4\}$

$$(B^{-1}N)^T = \begin{bmatrix} 1 & 0.2 & 1.6 \\ 0 & 0.2 & -0.4 \\ 0.5 & -0.1 & -0.3 \\ 2 & -0.2 & -1.6 \end{bmatrix}, \quad (B^{-1}N)^T e_3 = \begin{bmatrix} 0.2 \\ 0.2 \\ -0.1 \\ -0.2 \end{bmatrix}$$

$$z_N^* = \begin{bmatrix} 1.2 \\ 0.2 \\ 0.9 \\ 2.8 \end{bmatrix}, \quad 0 \le z_N^* + (B^{-1}N)^T z_B$$

$x_3 \ge -\frac{1.2}{0.2} = -6 \Rightarrow j=1$

$x_3 \ge -\frac{0.2}{0.2} = -1 \Rightarrow j=5$

$x_3 \le \frac{0.9}{0.1} = 9 \Rightarrow j=6 \leftarrow$ most constraining variable for the dual

switch x_3 & x_6

$x_3 \le \frac{2.8}{0.2} = 14 \Rightarrow j=4$

$B = \{2 \quad 6 \quad 7\}$ $\quad c^T = [1 \ 2 \ 1 \ 1 \ 0 \ 0 \ 0]$

$N = \{1 \quad 5 \quad 3 \quad 4\}$

$$A = \begin{bmatrix} 2 & 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 4 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 26 \\ 18 \end{bmatrix} \quad c_B = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad x_B^* = B^{-1} b = \begin{bmatrix} 8 \\ 10 \\ 10 \end{bmatrix}$$ optimal!

$$z^* = c_B^T B^{-1} b = 16$$

$(x_1, x_2, x_3, x_4) = (0, 8, 0, 0), \quad z^* = 16$

[C] #9.5

$$x_1 + 2x_2 + x_3 + x_4 \leq 5 \qquad x_3, x_4 \geq 0$$

$$3x_1 + x_2 - x_3 + 0x_4 \leq 8$$

check $x^* = (3, -1, 0, 2)$

$$x_2 + x_3 + x_4 = 1$$

$$3 + 2(-1) + 0 + 2 = 3 \leq 5 \checkmark$$

$$3(3) + (-1) - 0 + 0(2) = 8 \leq 8 \checkmark$$

$$-1 + 0 + 2 = 1$$

$x_3 = 0$, $x_4 = 2 \geq 0$, x^* is feasible

x_1: $y_1 + 3y_2 = 6$

x_2: $2y_1 + y_2 + y_3 = 1$

$y_1 - y_2 + y_3 \leq -1$

$y_1 + y_3 \leq -1$

First constraint slack:

$5 - (3) = 2 > 0 \Rightarrow y_1 = 0$

$x_4 = 2 > 0$, so reduced cost

$y_1 + y_3 = -1$

so, $y_1 = 0$, $3y_2 = 6 \Rightarrow y_2 = 2$

$2y_1 + y_2 + y_3 = 1 \Rightarrow y_3 = 1 - 2 \Rightarrow y_3 = -1$

$y_1 - y_2 + y_3 = 0 - 2 - 1 = -3 \leq -1 \checkmark$

$y_1 = 0 \geq 0$, $y_2 = 2 \geq 0 \checkmark$

$(y_1, y_2, y_3) = (0, 2, -1)$,

Primal value at x^*: $6x_1 + x_2 - x_3 - x_4 = 6 \cdot 3 + (-1) - 0 - 2 = 15$

Dual value: $5y_1 + 8y_2 + 1 \cdot y_3 = 5 \cdot 0 + 8 \cdot 2 + (-1) = 15$

Primal & dual feasible $\Rightarrow$ zero duality gap $\Rightarrow$ optimal.

so $x^* = (3,-1,0,2)$ is optimal

3. [v] $\quad x^* = (2,0,1)$, $z^* = 13$, $y^* = (1,0,1)$

New : $3x_1 + x_2 - x_3 \leq 3$

$$\min \quad 5y_1 + 11y_2 + 8y_3 + 3y_4$$

$$\begin{aligned} 2y_1 + 4y_2 + 3y_3 + 3y_4 &\geq 5 \qquad y \geq 0 \\ 3y_1 + y_2 + 4y_3 + y_4 &\geq 4 \\ y_1 + 2y_2 + 2y_3 - y_4 &\geq 3 \end{aligned}$$

$$\begin{cases} 3y_3 + 3y_4 = 5 \\ 2y_3 - y_4 = 3 \end{cases} , \quad y_3, y_4 \geq 0 \Rightarrow y_3 = \frac{14}{9}, \ y_4 = \frac{1}{9}$$

$$\begin{cases} 3x_1 + 2x_3 = 8 \\ 3x_1 - x_3 = 3 \\ x_2 = 0 \end{cases} \Rightarrow x_1 = \frac{14}{9}, \ x_3 = \frac{5}{3}, \ x_2 = 0$$

obj: $5 \cdot \frac{14}{9} + 3 \cdot \frac{5}{3} = \frac{115}{9}$

$x = \left(\frac{14}{9}, 0, \frac{5}{3}\right)$, $z^* = \frac{115}{9}$