5.1

$\max\ z = x_1 - 2x_2$

$x_1 + 2x_2 - x_3 + x_4 \geq 0$

$4x_1 + 3x_2 + 4x_3 - 2x_4 \leq 3$

$-x_1 - x_2 + 2x_3 + x_4 = 1$

dual vars $y_1, y_2, y_3 \Rightarrow$ free

$\min\ w = 0 \cdot y_1 + 3 \cdot y_2 + 1 \cdot y_3 = 3y_2 + y_3$

col of $x_1 : (1, 4, -1)^T$,

$1 \cdot y_1 + 4 \cdot y_2 - 1 \cdot y_3 = c_{x_1} = 1$

col of $x_2 : (2, 3, -1)^T$, $2y_1 + 3y_2 - y_3 \geq c_{x_2} = -2$

col of $x_3 : (-1, 4, 2)^T$, $-y_1 + 4y_2 + 2y_3 \geq c_{x_3} = 0$

Col of $x_4 : (1, -2, 1)^T$, $y_1 - 2y_2 + y_3 = c_{x_4} = 0$

$\min\ 3y_2 + y_3$

$y_1 + 4y_2 - y_3 = 1$ $\qquad y_1 \leq 0,\ y_2 \geq 0,\ y_3$ free

$2y_1 + 3y_2 - y_3 \geq -2$

$-y_1 + 4y_2 + 2y_3 \geq 0$

$y_1 - 2y_2 + y_3 = 0$

#5.6 $\max \; z = -x_1 - 2x_2$

$$-2x_1 + 7x_2 \le 6 \qquad x_1, x_2 \ge 0$$
$$-3x_1 + x_2 \le -1$$
$$9x_1 - 4x_2 \le 6$$
$$x_1 - x_2 \le 1$$
$$7x_1 - 3x_2 \le 6$$
$$-5x_1 + 2x_2 \le -3$$

check $x_2 = 0$, $-2x_1 \le 6 \Rightarrow$ no effect

$$-3x_1 \le -1 \Rightarrow x_1 \ge 1/3$$
$$9x_1 \le 6 \Rightarrow x_1 \le 2/3$$
$$x_1 \le 1 \Rightarrow x_1 \le 1$$
$$7x_1 \le 6 \Rightarrow x_1 \le 6/7$$
$$-5x_1 \le -3 \Rightarrow x_1 \ge 3/5$$

$$x_1 \in \left\{ \max\{1/3, 3/5, 0\}, \min\{2/3, 1, 6/7\} \right] = \left\{ \frac{3}{5}, \frac{2}{3} \right\}$$

on x_2, want $z = -x_1$, pick $x_1 = \frac{3}{5}$

$$(x_1^*, x_2^*) = \left(\frac{3}{5}, 0\right)$$

It checks out ~~for~~ feasibility

$$z^* = -x_1^* - 2x_2^* = -\frac{3}{5} - 0 = -\frac{3}{5}$$

$$(x_1^*, x_2^*) = \left(\frac{3}{5}, 0\right)$$

2.10

$\max\ 6x_1 + 8x_2 + 5x_3 + 9x_4$

$x_1 + x_2 + x_3 + x_4$, $x_{1,2,3,4} \geq 0$

$A = [1\ 1\ 1\ 1]$, $b = 1$, $c = (6, 8, 5, 9)^T$

$A^T = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $y \geq 6$, $y \geq 8$, $y \geq 5$, $y \geq 9$

feasible set is $y \geq 9$

$A^T y^* - c = (9-6, 9-8, 9-5, 9-9) = (3, 1, 4, 0)$

$x^* = (0,0,0,1)$, optimum $= 9$

3.4 $A \in \mathbb{R}^{m \times n}$, $b = 0$

Primal, $\max c^T x$ s.t $Ax \le 0$, $x \ge 0$ (P)

Dual, $\min b^t y$ s.t $A^T y \ge c$, $y \ge 0$ (D)

$\min 0$ s.t $A^T y \ge c$, $y \ge 0$

CASE: (D) is feasible

Pick a feasible y, by weak duality, $c^T x \le 0$ for every feasible x. But $x = 0$ is feasible and attains $c^T x = 0$

$\underline{x = 0 \text{ is optimal.}}$

Case: (D) is infeasible

Primal is feasible. If (P) were bounded above, duality $\Rightarrow$ (D) is feasible and has some finite optimum. ouch ↯

$\underline{\text{(P) must be unbounded}}$

1.2

$$\min\ -8x_1+9x_2+2x_3-6x_4-5x_5$$

$$6x_1+6x_2-10x_3+2x_4-8x_5 \geq 3\ ,\ x \geq 0$$

$$-6x_1-6x_2+10x_3-2x_4+8x_5 \leq -3$$

$$\max\ 8x_1-9x_2-2x_3+6x_4+5x_5$$

$$-6x_1-6x_2+10x_3-2x_4+8x_5 \leq -3 \qquad x \geq 0$$

$$A^T=(6,6,-10,2,-8)^T,\ c=(-8,9,2,-6,-5)^T$$

$$\max\ 3y \quad \begin{cases} 6y \leq -8 \\ 6y \leq 9 \\ -10y \leq 2 \\ 2y \leq -6 \\ -8y \leq -5 \\ y \geq 0 \end{cases}$$

$6y \leq -8$ contradicts $y \geq 0 \Rightarrow$ dual is infeasible

Primal is feasible but dual is infeasible

$\Longrightarrow$ primal unbounded

#5.3

$$\max \quad c^T x = 7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5$$

$Ax \le b, \ x \ge 0$,

$$A = \begin{pmatrix} 1 & 3 & 5 & -2 & 2 \\ 4 & 2 & -2 & 1 & 1 \\ 2 & 4 & 4 & -2 & 5 \\ 3 & 1 & 2 & -1 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 3 \\ 5 \\ 1 \end{pmatrix}$$

$$x^* = (0, 4/3, 2/3, 5/3, 0)$$

$$Ax^* = (4, 3, 14/3, 1)^T \le (4, 3, 5, 1)^T = b$$

$$\begin{cases} 3y_1 + 2y_2 + y_4 = 6 \\ 5y_1 - 2y_2 + 2y_4 = 5 \\ -2y_1 + y_2 - y_4 = -2 \end{cases}$$

$$y^* = (y_1, y_2, y_3, y_4) = (1, 1, 0, 1)$$

$$A^T y^* = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \\ 5 & -2 & 4 & 2 \\ -2 & 1 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 5 \\ -2 \\ 1 \end{pmatrix}$$

$$c = (7, 6, 5, -2, 3)^T$$

increasing x_5 would improve the objective, contradicting optimality

4.

(a) $(x_1^*, x_2^*) = (25, 75)$

$x_1 + x_2 = 100$, $10x_1 + 50x_2 = 4000$

$s_1, s_2 \geq 0$ 0 at opt. $\{x_1, x_2\}$ is basic set

$$\begin{cases} x_1 + x_2 + s_1 = 100 \\ 10x_1 + 50x_2 + s_2 = 4000 \end{cases}, \quad B = \begin{bmatrix} 1 & 1 \\ 10 & 50 \end{bmatrix}, \quad B^{-1} = \frac{1}{40}\begin{bmatrix} 50 & -1 \\ -10 & 1 \end{bmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = B^{-1}\left(\begin{pmatrix} 100 \\ 4000 \end{pmatrix} - \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}\right) = \begin{cases} x_1 = 25 - \frac{5}{4}s_1 + \frac{1}{40}s_2 \\ x_2 = 75 + \frac{1}{4}s_1 - \frac{1}{40}s_2 \end{cases}$$

$$z = 40x_1 + 70x_2 = 6250 - 32.5s_1 - 0.75s_2$$

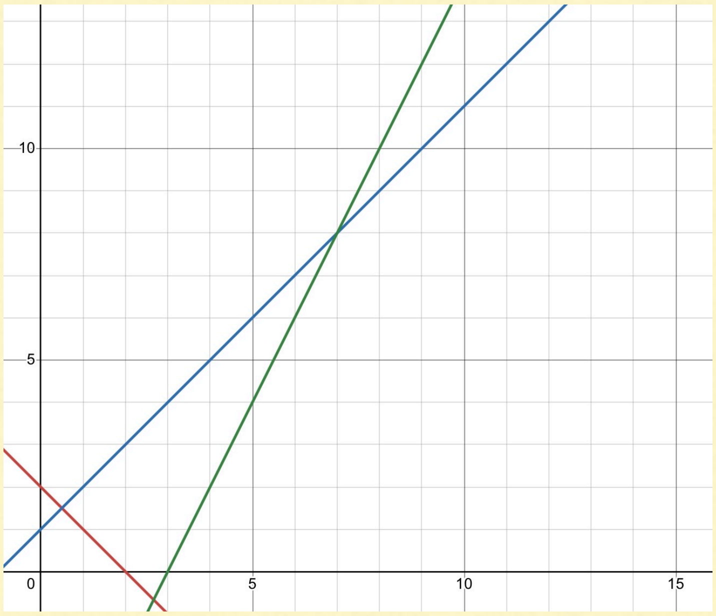
$$\begin{cases} x_1 = 25 - \frac{5}{4}s_1 + \frac{1}{40}s_2 \\ x_2 = 75 + \frac{1}{4}s_1 - \frac{1}{40}s_2 \\ z = 6250 - 32.5s_1 - 0.75s_2 \end{cases}$$

$s_1 = s_2 = 0 \Rightarrow x_1 = 25, x_2 = 75, z = 6250$

b) $A = \begin{bmatrix} 1 & 1 \\ 10 & 50 \end{bmatrix}$, $b = (100, 4000)^T$

$y^* = (32.5, 0.75)$

5. a)



b) $\min x_0$, $x_1 + x_2 - x_0 \leq 2$

$x_1 - x_2 - x_0 \leq -1$

$-2x_1 + x_2 - x_0 \leq -6$

$-x_1 - x_0 \leq$, $-x_2 - x_0 \leq 0$

$x_1, x_2, x_0 \geq 0$

Phase I: $x_1 + x_2 - x_0 = 2$

$x_1 - x_2 - x_0 = 1$

$-2x_1 + x_2 - x_0 = -6$

$2x_1 - 2x_0 = 1 \Rightarrow x_1 = \frac{1}{2} + x_0$

$\frac{1}{2} + x_0 + x_2 - x_0 = 2 \Rightarrow x_2 = \frac{3}{2}$

$-2\left(\frac{1}{2} + x_0\right) + \frac{3}{2} - x_0 = -6 \Rightarrow -1 - 2x_0 + \frac{3}{2} - x_0 = -6$

$\Rightarrow -\frac{1}{2} - 3x_0 = -6 \Rightarrow x_0 = \frac{13}{6}$

$x_1 = \frac{1}{2} + x_r = \frac{1}{2} + \frac{13}{6} = \frac{8}{3}$, $x_2 = \frac{3}{2}$, $x_0 = \frac{13}{6}$

$-x_1 - x_0 = -\frac{8}{3} - \frac{13}{6} \leq 0$, $-x_2 - x_0 = -\frac{3}{2} - \frac{13}{6} \leq 0$

$x_0^* = \frac{13}{6} > 0$

c) $\frac{1}{6}(1) + \frac{1}{2}(2) + \frac{1}{3}(3)$

LHS: $\frac{1}{6}(x_1 + x_2) + \frac{1}{2}(x_1 - x_2) + \frac{1}{3}(-2x_1 + x_2)$

$= \left(\frac{1}{6} + \frac{1}{2} - \frac{2}{3}\right)x_1 + \left(\frac{1}{6} - \frac{1}{2} + \frac{1}{3}\right)x_2 = 0$

RHS: $\frac{1}{6} \cdot 2 + \frac{1}{2}(-1) + \frac{1}{3}(-6) = \frac{1}{3} - \frac{1}{2} - 2 = -\frac{13}{6}$