4.1 $\max \quad 4x_1 + 5x_2$

$$2x_1 + 2x_2 \leq 9$$
$$x_1 \leq 4$$
$$x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

a)

i) $\max \; z = 4x_1 + 5x_2$

$$w_1 = 9 - 2x_1 - 2x_2 \qquad x_1, x_2, w_1, w_2, w_3 \geq 0$$
$$w_2 = 4 - x_1$$
$$w_3 = 3 - x_2$$

ii) Enter x_3, w_3 leaving:

$$z = 15 + 4x_1 - 5w_3 \qquad x_1, x_2, w_1, w_2, w_3 \geq 0$$
$$w_1 = 3 - 2x_1 + 2w_3$$
$$x_2 = 3 - w_3$$

iii) Enter x_1, w_1 leaving

$$z = 21 - 2w_1 - w_3$$
$$x_1 = 1.5 - 0.5w_1 + w_3$$
$$w_2 = 2.5 + 0.5w_1 - w_3 \qquad x_1, x_2, w_1, w_2, w_3 \geq 0$$
$$x_2 = 3 - w_3$$

No further

b) smallest - index

ii) x_1 enters, w_2 leaving

$$\max\ z = 16 - 4w_2 + 5x_2 \qquad x_1, x_2, w_1, w_2, w_3 \geq 0$$
$$w_1 = 1 + 2w_2 - 2x_2$$
$$x_1 = 4 - w_2$$
$$w_3 = 3 - x_2$$

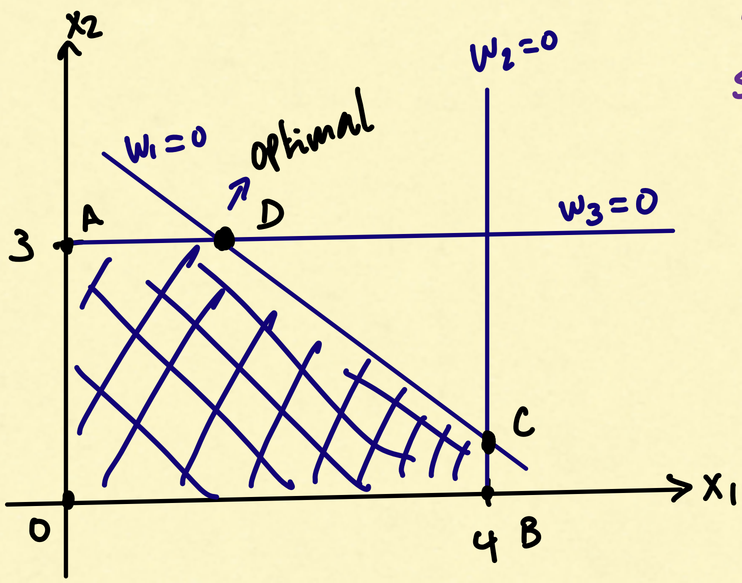
iii) x_2 enters, w_1 leaving

$$\max\ z = 18.5 + w_2 - 2.5w_1 \qquad x_1, x_2, w_1, w_2, w_3 \geq 0$$
$$x_2 = 0.5 + w_2 - 0.5w_1$$
$$x_1 = 4 - w_2$$
$$w_3 = 2.5 - w_2 + 0.5w_1$$

iv) w_2 enters, w_3 leaving

$$\max\ z = 25 - w_3 - 2w_1 \qquad x_1, x_2, w_1, w_2, w_3 \geq 0$$
$$x_2 = 3 - w_3$$
$$x_1 = 1.5 + w_3 - 0.5w_1$$
$$w_2 = 2.5 - w_3 + 0.5w_1$$

No further



Largest-coeff: O, A, D
smallest-ind: O, B, C, D

Ans: Largest-coeff pivoting rule was faster.
Largest-coeff: 2 iterations
smallest-ind: 3 iterations

4.2 max $2x_1 + x_2$ $\quad x_1, x_2 \geq 0$

$3x_1 + x_2 \leq 3$

i) max $z = 2x_1 + x_2$ $\quad x_1, x_2, w_1 \geq 0$

$w_1 = 3 - 3x_1 - x_2$

a) largest-coeff:

ii) x_1 enters, w_1 leaving

max $z = 2 - 0.67w_1 + 0.33x_2$ $\quad x_1, x_2, w_1 \geq 0$

$x_1 = 1 - 0.33w_1 - 0.33x_2$

iii) x_2 enters, x_1 leaving

max $z = 3 - w_1 - x_1$ $\quad x_1, x_2, w_1 \geq 0$

$x_2 = 3 - w_1 - 3x_1$

No further

b) smallest-index:

ii) x_1 enters, w_1 leaving

max $z = 2 - 0.67w_1 + 0.33x_2$ $\quad x_1, x_2, w_1 \geq 0$

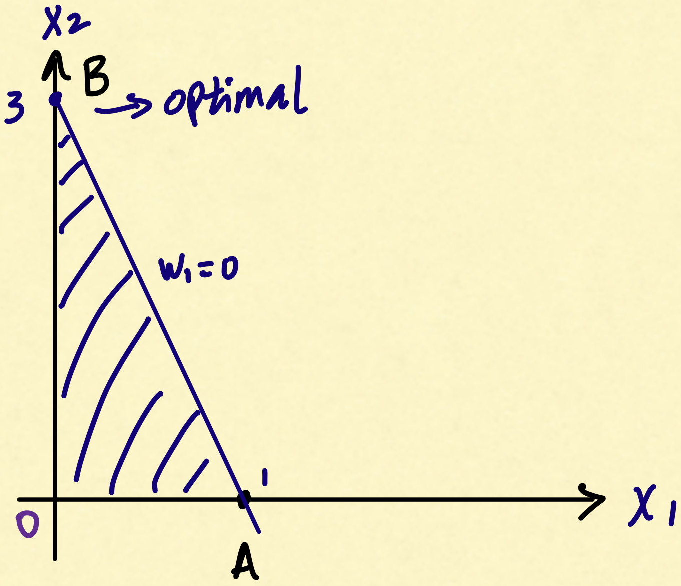
$x_1 = 1 - 0.33w_1 - 0.33x_2$

iii) x_2 enters, x_1 leaving

max $z = 3 - w_1 - x_1$ $\quad x_1, x_2, w_1 \geq 0$

$x_2 = 3 - w_1 - 3x_1$

No further



Largest-coeff: 0, A, B → 2 iterations

smallest-index: 0, A, B → 2 iterations

Ans: Both pivoting rules were equally fast.

4.3

$$\max \; 3x_1 + 5x_2$$

$$x_1 + 2x_2 \leq 5$$
$$x_1 \leq 3$$
$$x_2 \leq 2$$

$x_1, x_2 \geq 0$

i) $\max \; z = 3x_1 + 5x_2$

$w_1 = 5 - x_1 - 2x_2$

$w_2 = 3 - x_1$

$w_3 = 2 - x_2$

$x_1, x_2, w_1, w_2, w_3 \geq 0$

a) largest-coeff

ii) x_2 enters, w_3 leaving

$z = 10 + 3x_1 - 5w_3$

$w_1 = 1 - x_1 + 2w_3$

$w_2 = 3 - x_1$

$x_2 = 2 - w_3$

$x_1, x_2, w_1, w_2, w_3 \geq 0$

iii) x_1 enters, w_1 leaving

$\max \; z = 13 - 3w_1 + w_3$

$w_1 = 1 - w_1 + 2w_3$

$w_2 = 2 + w_1 - 2w_3$

$x_2 = 2 - w_3$

$x_1, x_2, w_1, w_2, w_3 \geq 0$

iv) w_3 enters, w_2 leaving

$$\max z = 14 - \tfrac{5}{2}w_1 - \tfrac{1}{2}w_2 \qquad x_1, x_2, w_1, w_2, w_3 \geq 0$$

$$w_1 = 3 \qquad - w_2$$

$$w_2 = 1 + \tfrac{1}{2}w_1 - \tfrac{1}{2}w_2$$

$$x_2 = 1 - \tfrac{1}{2}w_1 + \tfrac{1}{2}w_2$$

No further

b) smallest-index

ii) x_1 enters, w_2 leaving

$$\max z = 9 - 3w_2 + 5x_2 \qquad x_1, x_2, w_1, w_2, w_3 \geq 0$$

$$w_1 = 2 + w_2 - 2x_2$$

$$x_1 = 3 - w_2$$

$$w_3 = 2 \qquad - w_3$$

iii) x_2 enters, w_1 leaving

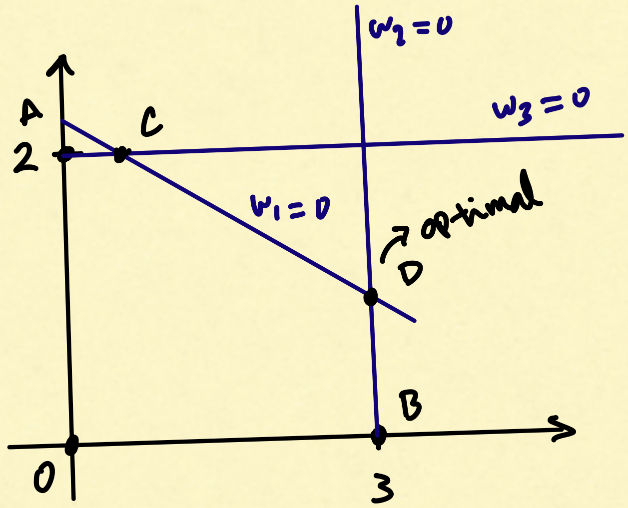
$$\max z = 14 - \tfrac{1}{2}w_2 - \tfrac{5}{2}w_1 \qquad x_1, x_2, w_1, w_2, w_3 \geq 0$$

$$x_2 = 1 + \tfrac{1}{2}w_2 - \tfrac{1}{2}w_1$$

$$x_1 = 3 - w_2$$

$$w_3 = 1 - \tfrac{1}{2}w_2 + \tfrac{1}{2}w_1$$

No further



Largest-coeff: 0, A, C, D 3 iterations

smallest-index: 0, B, D 2 iterations

<u>Ans</u>: smallest-index pivoting rule was faster

(2).

a)

$$f(\lambda X + (1-\lambda) Y) = A(\lambda X + (1-\lambda) Y) + b$$
$$= \lambda AX + (1-\lambda) AY + b$$
$$= \lambda(AX+b) + (1-\lambda)(AY+b)$$
$$= \lambda f(X) + (1-\lambda) f(Y)$$
$$\leq \lambda f(x) + (1-\lambda) f(y) \checkmark$$

Not strictly convex:

$\lambda = 2$, $f(2X - y) = 2f(x) - f(y) \leq 2f(x) - f(y)$

b) Suppose that $X^*, y^* \in \mathbb{R}^n$ and that they are minimizers of a convex f.

so, $f(X^*) = f(y^*) = m = \inf\limits_{X} f(X)$

$f(\lambda X^* + (1-\lambda) Y^*) \leq \lambda f(X^*) + (1-\lambda) f(Y^*) = \lambda m + (1-\lambda) m = m$

but m is the minimum value, so $f(\lambda X^* + (1-\lambda) Y^*) \geq m$

$f(\lambda X^* + (1-\lambda) Y^*) = m$

so, $\{\lambda : \lambda X^* + (1-\lambda) Y^*\}$

c) Suppose that f is strictly convex and that there are two distinct minimizers $x^* = y^*$ w/ $f(x^*) = f(y^*) = m$ for any $\lambda \in \{0,1\}$

so by strict convexity we have,

$$f(\lambda x^* + (1-\lambda)y^*) < \lambda f(x^*) + (1-\lambda)f(y^*) = m$$

but this contradicts minimality of m.

there can't be two minimizers, ouch $\lightning$

must be unique!

3.

$$\max\ 4x_1 + 2x_2 - x_3 + 2x_4$$

$$\text{s.t}\quad x_1 + x_2 + 3x_3 + 4x_4 = 8$$

$$x_1 + x_2 + x_3 + x_4 = 4$$

$$x_1, x_2, x_3, x_4 \geq 0$$

m1) $x_4 = \dfrac{4-2x_3}{3}$, $x_1 = \dfrac{8-3x_2-x_3}{3}$, feasible when $x_2, x_3 \geq 0$, $x_3 \leq 2$

$$z = 4x_1 + 2x_2 - x_3 + 2x_4 = \frac{40}{3} - 2x_2 - \frac{11}{3}x_3$$

reduced cost of $x_2, x_3 \leq 0$, $x_2 = x_3 = 0$ ← initial

so already optimal, $(x_1, x_2, x_3, x_4) = (\frac{8}{3}, 0, 0, \frac{4}{3})$

and $z^* = 40/3$

m2)

$$x_1 + x_2 + 3x_3 + 4x_4 \leq 8 \quad (1) \qquad x_{1,2,3,4} \geq 0$$

$$x_1 + x_2 + x_3 + x_4 \leq 4 \quad (2)$$

$$x_1 + x_2 + 3x_3 + 4x_4 + s_1 = 8$$

$$x_1 + x_2 + x_3 + x_4 + s_2 = 4$$

Phase I:

$$x_1 + x_2 + 3x_3 + 4x_4 + s_1 - x_0 = 8$$

$$x_1 + x_2 + x_3 + x_4 + s_2 - x_0 = 4$$

$$\max\ -x_0$$

take $x_1 = x_2 = x_3 = x_4 = x_0 = 0$, $s_1 = 8$, $s_2 = 4$

we get $s_1 = 8 - x_1 - x_2 - 3x_3 - 4x_4$

$s_2 = 4 - x_1 - x_2 - x_3 - x_4$

feasible dict.

Phase II: $z = 4x_1 + 2x_2 - x_3 + 2x_4$

Pivot x_1.

$$s_1 = 4 - 2x_3 - 3x_4 + s_2$$

$$z = 16 - 2x_2 - 5x_3 - 2x_4 - 4s_2$$

$$x_2 = x_3 = x_4 = s_2 = 0, \quad x_1 = 4, \quad s_1 = 4, \quad z = 16$$

$$x_4 = \frac{4 - 2x_3}{3}, \quad x_1 = \frac{8 - 3x_2 - x_3}{3}$$

$$z = \frac{40}{3} - 2x_2 - \frac{11}{3}x_3$$

$$x_2 = x_3 = 0$$

$$\left(\frac{8}{3}, 0, 0, \frac{4}{3}\right) \quad z^* = \frac{40}{3}$$

m3) $x_0 \geq 0$, $\quad x_1 + x_2 + 3x_3 + 4x_4 - x_0 = 8$

$\quad x_1 + x_2 + x_3 + x_4 - x_0 = 4$

take $(x_1, x_2, x_3, x_4) = (0, 0, 0, \frac{4}{3})$ & $x_0 = 0$

$4x_4 = \frac{16}{3}$ (1st row), $x_4 = \frac{4}{3}$ (2nd row)

$$x_4 = \frac{4 - 2x_3}{3}, \quad x_1 = \frac{8 - 3x_2 - x_3}{3}, \quad x_0 = 0$$

$$z = 4x_1 + 2x_2 - x_3 + 2x_4 = \frac{40}{3} - 2x_2 - \frac{11}{3}x_3$$

$$\left(\frac{8}{3}, 0, 0, \frac{4}{3}\right), \quad z^* = \frac{40}{3}$$