p. 22 $\quad \max \; 6x_1 + 8x_2 + 5x_3 + 9x_4$

2.1

$$2x_1 + x_2 + x_3 + 3x_4 \leq 5$$
$$x_1 + 3x_2 + x_3 + 2x_4 \leq 3$$
$$x_1, x_2, x_3, x_4 \geq 0$$

slack vars, s_1, s_2

$$s_1 = 5 - 2x_1 - x_2 - x_3 - 3x_4$$
$$\star\; s_2 = 3 - x_1 - 3x_2 - x_3 - 2x_4$$
$$\zeta = 6x_1 + 8x_2 + 5x_3 + 9x_4$$

pick x_4: largest non positive ratio $-\frac{3}{2} > -\frac{5}{3}$

$$\star \Rightarrow x_4 = \frac{3}{2} - \frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}s_2$$

$$s_1 = 5 - 2x_1 - x_2 - x_3 - 3\left(\frac{3}{2} - \frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}s_2\right)$$
$$= \frac{1}{2} - \frac{1}{2}x_1 + \frac{7}{2}x_2 + \frac{1}{3}x_3 + \frac{3}{2}s_2$$
$$\zeta = \frac{27}{2} + \frac{3}{2}x_1 - \frac{11}{2}x_2 + \frac{1}{2}x_3 - \frac{9}{2}s_2$$

② pick x_1: pick $-\frac{1}{2}x_1$ in s_1 $\quad \frac{1/2}{1/2} < \frac{3/2}{1/2}$

$$\begin{cases} x_1 = 2 + 7x_2 + x_3 - 2s_1 + 3s_2 \\ x_4 = 1 - 5x_2 - x_3 + s_1 - 2s_2 \\ \zeta = 15 + 5x_2 + 2x_3 - 3s_1 \end{cases}$$

③ pick x_2: pick $-\frac{1}{5}$ in x_4

$$\begin{cases} x_2 = \frac{1}{5} - \frac{1}{5}x_3 - \frac{1}{5}x_4 + \frac{1}{5}s_1 - \frac{2}{5}s_2 \\ x_1 = \frac{12}{5} - \frac{2}{5}x_3 - \frac{7}{5}x_4 - \frac{3}{5}s_1 + \frac{1}{5}s_2 \\ \zeta = 16 + x_3 - x_4 - 2s_1 - 2s_2 \end{cases}$$

③: pick x_3, $-\frac{1/5}{1/5} > -\frac{12/5}{2/5}$, pick eqn x_2

$$x_3 = 1 - 5x_2 - x_4 + s_1 - 2s_2$$
$$x_1 = 2 + 2x_2 - x_4 - s_1 + s_2$$
$$\zeta = 17 - 5x_2 - 2x_4 - s_1 - 4s_2$$

so, $x_2 = x_4 = s_1 = s_2 = 0 \Rightarrow x_1 = 2,\ x_3 = 1$

$\zeta_{max} = 17$, soln: $(2, 0, 1, 0, 0, 0)$

#2.2 max $2x_1 + x_2$ pick x_1,

$$2x_1 + x_2 \le 4 \to \frac{4}{2}$$
$$2x_1 + 3x_2 \le 3 \to \frac{3}{2}$$
$$4x_1 + x_2 \le 5 \to \frac{5}{4}$$
$$x_1 + 5x_2 \le 1 \to \frac{1}{1} \checkmark \text{ pick } s_4$$
$$x_1, x_2 \ge 0$$

$$x_1 = 1 - 5x_2 - s_4$$

$\Rightarrow s_1 = 2 + 9x_2 + 2s_4$ $\quad \zeta = 2 - 9x_2 - 2s_4$

$s_2 = 1 + 7x_2 + 2s_4$ $\quad (x_1 = 1, x_2 = 0)$

$s_3 = 1 + 19x_2 + 4s_4$

2.5 phase I: $w = a_1 + a_2$

$$a_1 = 3 - x_1 - x_2 + s_1$$
$$a_2 = 1 - x_1 + x_2 + s_2$$
$$s_3 = 4 - x_1 - 2x_2$$
$$w = 4 - 2x_1 - 0x_2 + s_1 + s_2$$

$a_1 \geq 0 \Rightarrow x_1 \leq 3$, $a_2 \geq 0 \Rightarrow x_1 \leq 1$, $s_3 \geq 0 \Rightarrow x_1 \leq 4$

$a_2 = 1 - x_1 + x_2 + s_2 \Rightarrow x_1 = 1 + x_2 + s_2 - a_2$

$$a_1 = 2 - 2x_2 - a_2 + s_1 - s_2$$
$$s_3 = 3 - 3x_2 + a_2 - s_2$$
$$w = 2 - 2x_2 - s_2 + 2a_2 + s_1$$

$a_1 \geq 0 \Rightarrow x_2 \leq 1$, $s_3 \geq 0 \Rightarrow x_2 \leq 1$

$s_3 \Rightarrow x_2 = 1 + \frac{1}{3}a_2 - \frac{1}{3}s^2 - \frac{1}{3}s_3$

$x_1 = 2$, $x_2 = 1$, $a_1 = 0$, $w = 0$

plase 1 ends w/ feasible basis $x = (2,1)$

Phase II: max $z = x_1 + 3x_2$

$x_1 = 2 + \frac{2}{3}s_2 - \frac{1}{3}s_3$, $x_2 = 1 - \frac{1}{3}s_2 - \frac{1}{3}s_3$

$s_1 = x_1 + x_2 - 3 = \frac{1}{3}s_2 - \frac{2}{3}s_3 \geq 0 \Rightarrow s_2 \geq 2s_3$

$z = x_1 + 3x_2 = 5 - \frac{1}{3}s_2 - \frac{4}{3}s_3$

$s_2 = s_3 = 0$, $x_1 = 2$, $x_2 = 1$

2.6

Same as 2.5 but with $x_1 + 2x_2 + s_3 = 2$

so pivot x_1 in a_2 (same as before)

then pivot x_2 in using row 3.

Phase I $\Rightarrow$ $w = \frac{4}{3} + \frac{4}{3}a_2 + s_1 - \frac{1}{3}s_2 + \frac{2}{3}s_3$

so $w = \frac{4}{3} > 0$ cannot ferther reduce w to 0.

Infeasible.

2.10

$$\max\ z = 6x_1 + 8x_2 + 5x_3 + 9x_4$$

$$x_1 + x_2 + x_3 + x_4 = 1\ ,\ x \geq 0$$

$$x_4 = 1 - x_1 - x_2 - x_3$$

$$z = 9 - 3x_1 - x_2 - 4x_3$$

$$x_4 = 1,\ x_1 = x_2 = x_3 = 0,\ z = 9$$

3.4 $\max\ c^T x$ s.t $Ax \le 0,\ x \ge 0$

Let $S = \{ x \in \mathbb{R}^n : Ax \le 0,\ x \ge 0 \}$

so if $x \in S$ and $t \ge 0$

$\Rightarrow A(tx) = t(Ax) \le 0,\ tx \ge 0$

so $tx \in S$.

so S is polyhedral cone and $0 \in S$

Case 1: there is $x' \in S$ w/ $c^T x' > 0$

so for every $t \ge 0$, $tx' \in S$

and $c^T(tx') = t\,c^T x' \longrightarrow \infty$ as $t \to \infty$

obj. is unbounded above S. ouch $\lightning$

Case 2: For all $x \in S$, $c^T x \le 0$

then $\sup_{x \in S} c^T x \le 0$

but $0 \in S$ & $c^T 0 = 0$, so max val. must be 0 at $x = 0$.

So it is optimal!

#1.2 $\min -8x_1 + 9x_2 + 2x_3 - 6x_4 - 5x_5$

$$6x_1 + 6x_2 - 10x_3 + 2x_4 - 8x_5 \geq 3$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

$$\max 8x_1 - 9x_2 - 2x_3 + 6x_4 + 5x_5$$

$$-6x_1 - 6x_2 + 10x_3 - 2x_4 + 8x_5 \leq -3$$

$$6x_1 + 6x_2 - 10x_3 + 2x_4 - 8x_5 - s_1 = 3$$

2.2 $\max\ z = 2x_1 + 3x_2 + 5x_3 + 4x_4$

$$\begin{cases} x_1 + 2x_2 + 3x_3 + x_4 \leq 5 \\ x_1 + x_2 + 2x_3 + 3x_4 \leq 3 \end{cases} \qquad x_1, x_2, x_3, x_4 \geq 0$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 + x_4 + s_1 = 5 \\ x_1 + x_2 + 2x_3 + 3x_4 + s_2 = 3 \end{cases}$$

① pick x_3 and eqn s_2

$$\begin{cases} x_3 = \frac{3}{2} - \frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{3}{2}x_4 - \frac{1}{2}s_2 \\ -1 = -2s_1 + 3s_2 + x_1 - x_2 + 7x_4 = -1 \\ z = \frac{15}{2} - \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{7}{2}x_4 - \frac{5}{2}s_2 \end{cases}$$

② pick x_2 w/ eqn s_1

$$x_2 = 1 + x_1 + 7x_4 + 3s_2 - 2s_1$$

elim x_2, $x_3 = 1 - x_1 - 5x_4 + s_1 - 2s_2$

$$\zeta = 8 - s_1 - s_2$$

max at $s_1 = s_2 = 0$

$$\begin{cases} x_2 = 1 + x_1 + 7x_4 \\ x_3 = 1 - x_1 - 5x_4 \\ x_1 \geq 0,\ x \geq 4,\ x_1 + 5x_4 \leq 1 \\ \zeta = 8 \end{cases}$$

All feasible pts are optimal bc,

$(2,3,5,4) = (1,2,3,1) + (1,1,2,3)$ & $5+3=8$

so, $(0, \frac{12}{5}, 0, \frac{1}{5})$, $(1,2,0,0)$, $(0,1,1,0)$ are optimal

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3.
$$x_5 = -\tfrac{1}{2}x_1 + 5\tfrac{1}{2}x_2 + 2\tfrac{1}{2}x_3 - 9x_4$$

$$x_6 = -\tfrac{1}{2}x_1 + 1\tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 - x_4$$

$$x_7 = 1 - x_1$$

$$\zeta = 10x_1 - 57x_2 - 9x_3 - 24x_4$$

① pick x_1, x_5 leaves

$$x_1 = 11x_2 + 5x_3 - 18x_4 - 2x_5$$
$$x_6 = -4x_2 - 2x_3 + 8x_4 + x_5$$
$$x_7 = 1 - 11x_2 - 5x_3 + 18x_4 + 2x_5$$
$$\zeta = 53x_2 + 41x_3 - 204x_4 - 20x_5$$

② x_2, leave x_6,

$$x_2 = -\tfrac{1}{2}x_3 + 2x_4 + \tfrac{1}{4}x_5 - \tfrac{1}{4}x_6$$
$$x_1 = -\tfrac{1}{2}x_3 + 4x_4 + \tfrac{3}{4}x_5 - 2\tfrac{3}{4}x_6$$
$$x_7 = 1 + \tfrac{1}{2}x_3 - 4x_4 - \tfrac{3}{4}x_5 - 13\tfrac{1}{4}x_6$$
$$z = 14.5x_3 - 98x_4 - 6.75x_5 - 13.25x_6$$

③ x_3, leave x_1

$$x_3 = 8x_4 + 1.5x_5 - 5.5x_6 - 2x_1$$
$$x_2 = -2x_4 - 0.5x_5 + 2.5x_6 + x_1$$
$$x_7 = 1 - x_1 - 16x_6$$
$$z = 18x_4 + 15x_5 - 93x_6 - 29x_1$$

④ x_4, leave x_2

$$x_4 = -0.25x_5 + 1.25x_6 + 0.5x_1 - 0.5x_2$$
$$x_3 = -0.5x_5 + 4.5x_6 + 2x_1 - 4x_2$$
$$x_7 = 1 - x_1 - 16x_6$$
$$z = 10.5x_5 - 70.5x_6 - 20x_1 - 9x_2$$

⑤ x_5, leave x_3

$$x_5 = 9x_6 + 4x_1 - 8x_2 - 2x_3$$
$$x_4 = -x_6 - 0.5x_1 + 1.5x_2 + 0.5x_3$$
$$x_7 = 1 - x_1$$
$$z = 24x_6 + 22x_1 - 93x_2 - 21x_3$$

⑥ so the bland fix, picks x_1 and x_4 leaves

$x_4 = -x_6 - \tfrac{1}{2}x_1 + 1.5x_2 + 0.5x_3$ for x_1,

$$x_1 = -2x_4 - 2x_6 + 3x_2 + x_3$$
$$x_5 = x_6 + 4x_2 + 2x_3 - 8x_4$$
$$x_7 = 1 + 2x_4 - 3x_2 - x_3 - 14x_6$$
$$z = x_3 - 27x_2 - 20x_6 - 44x_4$$

⑦ x_3, x_7 leaves

$$x_3 = 1 + 2x_4 - 3x_2 - 14x_6 - x_7$$
$$x_1 = 1 - x_7 - 16x_6$$
$$x_5 = 2 - 2x_7 - 27x_6 - 2x_2 - 4x_2$$
$$z = 1 - x_7 - 42x_4 - 30x_2 - 34x_6$$

$$x_2 = x_4 = x_6 = x_7 = 0$$

$$x_1 = 1, x_3 = 1, x_5 = 2$$

#4.

$$x_4 = 3 + x_2 - 2x_5$$
$$x_1 = 1 - 5x_2 + 6x_5$$
$$x_6 = 4 + 9x_2 + 2x_5$$
$$z = 8 - x_3$$

$-x_2 + 2x_5 \leq 3$, $5x_2 - 6x_5 \leq 1$, $-9x_2 - 2x_5 \leq 4$,
$x_2, x_5 \geq 0$.

elim x_5 w/ $x_1 = 1 - 5x_2 + 6x_5 \Rightarrow x_5 = (x_1 - 1 + 5x_2)/6$

$x_1 \geq 0$, $x_2 \geq 0$, $x_1 + 2x_2 \leq 10$, $x_1 \geq 1 - 5x_2$, $x_3 = 0$

a) max $z = 8 - x_3$

$$x_1 + 2x_2 \leq 10$$
$$x_1 \geq 1 - 5x_2$$
$$x_1, x_2, x_3 \geq 0$$

b) optimal solution

$x_3 = 0$, $x_1 \geq 0$, $x_2 \geq 0$, $x_1 + 2x_2 \geq 10$, $x_1 \geq 1 - 5x_2$.