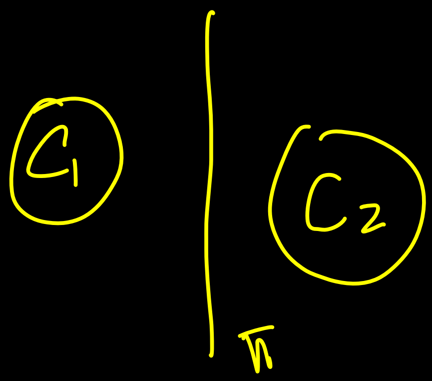
general theorem: $\mathbb{R}^n$

given 2 disjoint convex sets
C_1, C_2 - convex

there exists a half plane π, that can separate C_1, C_2

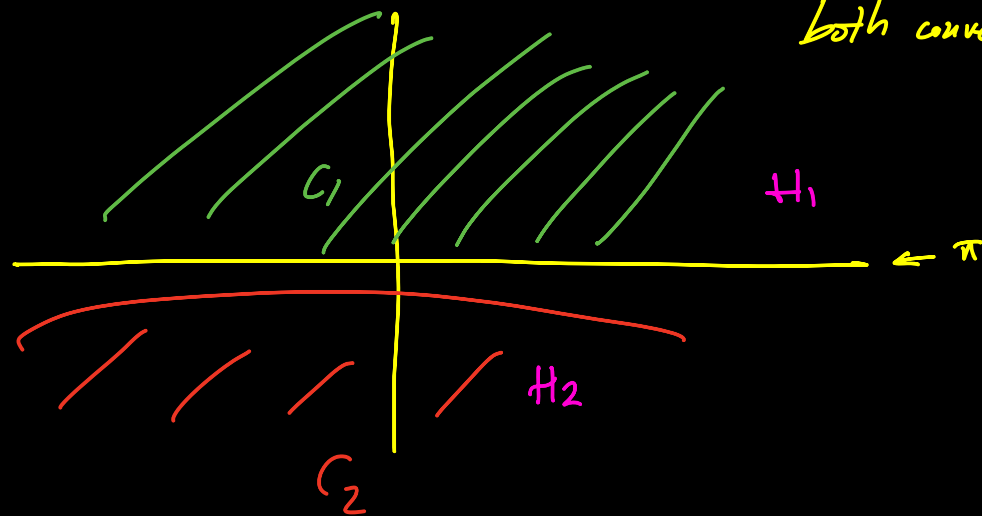


π separates $\mathbb{R}^n$ into two half spaces
s.t $C_1 \subseteq H_1$ and $C_2 \subseteq H_2$ H_1, H_2 open and closed sets.

both convex

$H_1 = \{y \geq 0\}$

$H_2 = \{y > 0\}$

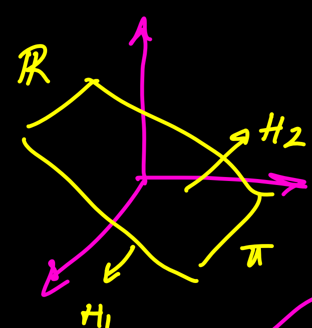


How to find π, given C_1 and C_2

Some terminology

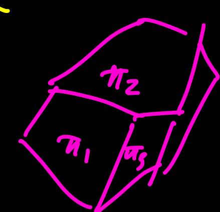
1) Half-plane: π $\quad a_1 x_1 + \dots + a_n x_n = b$

(2) Half-space: H $\quad a_1 x_1 + \dots + a_n x_n \leq b$



(3) Polyhedron = intersection between finitely many half-spaces (convex)

$$\boxed{a_{i1} x_1 + a_{i2} x_2 + \dots a_{in} x_n \leq b}$$



$\boxed{A\vec{x} \le \vec{b}}$ ← feasible set X
polyhedron.

$P = \{ X : Ax \le b \}$ is convex

$X_1, X_2 \in P$ i.e $AX_1 \le b,\ AX_2 \le b$

$\Rightarrow \underbrace{\lambda}_{\lambda_1} X_1 + \underbrace{(1-\lambda)}_{\lambda_2} X_2 \in P\ ?$ $\quad 0 \le \lambda \le 1$
$\quad 0 \le 1 - \lambda \le 1$

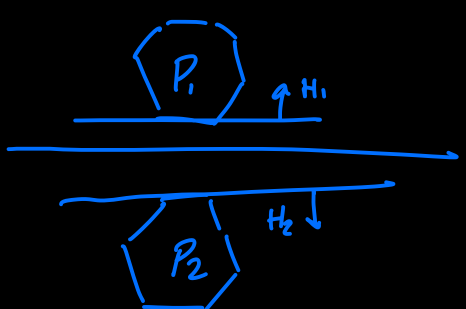
$\lambda_1 + \lambda_2 = 1, \quad \lambda_1, \lambda_2 \geq 0$

$$A(\lambda X_1 + (1-\lambda) X_2) = \lambda A X_1 + (1-\lambda) A X_2$$
$$\le \lambda b + (1-\lambda) b = b$$

[V] Thm 10.4

Let $P, \tilde{P}$ be 2 disjoint polyhedron.

Then there are 2 disjoint half spaces H_1, H_2 s.t
$P_1 \subseteq H_1,\ P_2 \subseteq H_2$



Can use LP
to find H_1, H_2 (Farkas' Lemma)

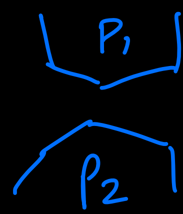
<u>Farkas Lemma [v]</u>

Consider the $AX \le b$ (*)

(*) has no solution $\iff$ there is a $\vec{y}$ s.t

$$y^T A = 0, \quad y \ge 0 \quad b^T y < 0$$

How to find H_1 and H_2 in Thm 10.4



$P_1 = \{X : AX \le b\}$, $P_2 = \{X : \tilde{A} X \le \tilde{b}\}$

$P_1 \cap P_2 = \emptyset$ $\qquad \left.\begin{matrix} Ax \le b \\ \tilde{A} x \le \tilde{b} \end{matrix}\right\} \leftarrow$ has no solution

$$FL \Longrightarrow \begin{bmatrix} A \\ \tilde{A} \end{bmatrix} [x] \le \begin{bmatrix} b \\ \tilde{b} \end{bmatrix} \leftarrow \text{no solution}$$

$y = (\ )$, $y^T = (\qquad)$

there are $\begin{pmatrix} y \\ \tilde{y} \end{pmatrix}$ s.t $(y \quad \tilde{y}) \begin{pmatrix} A \\ \tilde{A} \end{pmatrix} = \vec{0}$

$$\boxed{y^T A + \tilde{y}^T \tilde{A} = 0}$$

and $(b^T \quad \tilde{b}^T) \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} < 0$ i.e $b^T y + \tilde{b}^T \tilde{y} < 0$

$y^T A + \tilde{y}^T \tilde{A} = 0$ $\boxed{b^T y + \tilde{b}^T \tilde{y} < 0}$

Define $H_1 = \{X : y^T A x \le y^T b\}$

$H_2 = \{X : \tilde{y}^T \tilde{A} x \le \tilde{y}^T \tilde{b}\}$

$y^T (A X) \le y^T b$ a number

$\leftrightarrows (\;) \leftrightarrow (\;)$

$P_1 \subseteq H_1$

$P_2 \subseteq H_2$

$\boxed{a_1 X_1 + \dots + a_n X_n \le b}$ ← half space

H_1 equation

P_1 X: $y^T (AX \le b) \Rightarrow \underline{y^T A X \le y^T b}$ $P_1 \subseteq H_1$

P_2 X: $\tilde{y}^T (\tilde{A} x \le \tilde{b}) \Rightarrow \underline{\tilde{y}^T \tilde{A} X \le \tilde{y}^T \tilde{b}}$ ← H_2 equation $P_2 \subseteq H_2$

$(1 \;\; 2) \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4 + 6 = 10$

$(4 \;\; 3) \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 10$ bc just numbers

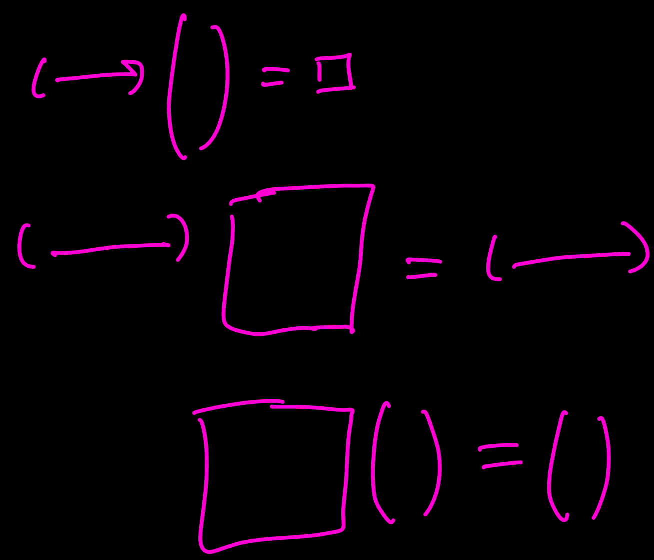
$(y^T A + \tilde{y}^T \tilde{A}) X \boxed{\le y^T b + \tilde{y}^T \tilde{b}}$ same

< 0 and $0 < 0$ ↯

Pf of FL:

$b^T y$ = a number

row   col



$Ax \le b$ (*)

"⇐" Suppose there is y s.t $y^T A = 0$ and $y^T b < 0$

Claim (*) has no solution

Suppose there is X s.t $Ax \le b$

$\underline{\underline{y \ge 0}}$ $y^T(Ax \le b)$

$0 \quad \cancel{y^T A} x \le y^T b < 0 \Rightarrow 0 < 0$ ↯

"⇒" Suppose (*) has no solution

$0x_1 + \dots + 0x_n$ then there is y satisfies ... (*)

max 0
s.t $Ax \le b$

(*) D
min $\mathfrak{z}(y) = b^T y$
s.t $yA^T = 0$
$y \ge 0$

max $f(x) = 5$

max over empty $= -\infty$

consider $(D)_*$ $\leftarrow$ always feasible

$y=0$ satisfy constraint

(P) $\underline{(D)}$

$z^* = \min z(y) \leq z(0) = 0$

Suppose $\cancel{z^*=0}$ $\Rightarrow$ opt $(D)_*$ has an opt soln

$\Rightarrow$ (P) also has opt soln (feasible)

$\Rightarrow z^* < 0 \overset{?}{\Longrightarrow}$ there is y s.t $\boxed{y^TA = 0 \text{ and } b^Ty < 0}$

$\Rightarrow z^* = -\infty$

$z(y) = b^Ty < 0$

$y^TA = 0$

$y \geq 0$

$y \to ty$, $t \geq 0$

$ty \geq 0$, $(ty)^TA = 0$ $b^T(ty) = Ab^Ty$