$$\max \ \zeta = -2x_1 - x_2 \qquad x_1, x_2 \geq 0$$

$$\text{s.t} \quad \left.\begin{aligned} -x_1 + x_2 &\leq -1 \\ -x_1 - 2x_2 &\leq -2 \\ x_2 &\leq 1 \end{aligned}\right\} \text{ origin is not feasible}$$

Auxiliary Problem

$$\max \ \zeta = -x_0$$

$$\text{s.t} \quad \begin{aligned} -x_1 + x_2 - x_0 &\leq -1 \\ -x_1 - 2x_2 - x_0 &\leq -2 \\ x_2 - x_0 &\leq 1 \end{aligned} \qquad x_1, x_2, x_0 \geq 0$$

(1) must be feasible :
choose x_0 large enough

(2) Original problem is feasible

$\Longleftrightarrow$ $\max \zeta = 0$ (Note : $\max \zeta \leq 0$)

$$\max \ \zeta = -x_0$$

$$\text{s.t} \quad \begin{aligned} w_1 &= -1 - x_1 - x_2 + x_0 \\ w_2 &= -2 + x_1 + 2x_2 + x_0 \\ w_3 &= 1 - x_2 + x_0 \end{aligned}$$

Set $x_1, x_2, x_0 = 0 \implies$ $w_1 = -1$ bc $w_1, w_2 < 0$

$w_2 = -2$ $\leftarrow$ most infeasible

(interchange x_0 & w_2)

$w_3 = 1$

$\max\ z = -x_0$

s.t. $w_1 = -1 - x_1 - x_2 + x_0$

$w_2 = -2 + x_1 + 2x_2 + x_0$

$w_3 = 1 - x_2 + x_0$

$x_0 = w_2 + 2 - x_1 - 2x_2$

$\downarrow$ insert x_0 in w_1 and w_3 and simplify

$w_1 = 1 - 3x_2 + w_2$

$w_3 = 3 - x_1 - 3x_2 + w_2$

$z = -x_0$

I

$\max\ z = -2 + x_1 + \underline{2x_2} - w_2$

s.t. $x_0 = 2 - x_1 - 2x_2 + w_2$, $x_0 = 2 - 2x_2 \geq 0$

$w_1 = 1 - 3x_2 + w_2$, $w_1 = 1 - 3x_2 \geq 0$

$w_3 = 3 - x_1 - 3x_2 + w_2$, $w_3 = 3 - 3x_2 \geq 0$

Set $x_1, x_2, w_2 = 0 \implies \left.\begin{array}{l} x_0 = 2 \\ w_1 = 1 \\ w_3 = 3 \end{array}\right\}$ feasible set! bc basic vars ≥ 0

pick x_2 highest coeff

$x_0: -2x_2 \geq -2 \implies x_2 \leq 1$

$w_1: -3x_2 \geq -1 \implies x_2 \leq 1/3 \leftarrow$ pick tightest so interchange x_2, w_1

$w_3: -3x_2 \geq -3 \implies x_2 \leq 1$

$x_2 = \frac{1}{3} - \frac{w_1}{3} + \frac{w_2}{3}$ (sub into x_0, w_3)

$w_3 = 2 - x_1 + w_1$

$x_0 = \frac{4}{3} - x_1 + \frac{2}{3} w_1 + \frac{1}{3} w_2$

max $\quad z = -\frac{4}{3} + \underline{x_1} - \frac{2}{3} w_1 - \frac{w_2}{3}$

Increase

$x_2 = \frac{1}{3} - \frac{1}{3} w_1 + \frac{1}{3} w_2$

s.t $\quad w_3 = 2 - x_1 + w_1$

$x_0 = \frac{4}{3} - x_1 + \frac{2}{3} w_1 + \frac{1}{3} w_2$

x_2 : Does not have x_1

w_3: $\quad w_3 = 2 - x_1 \geq 0 \Rightarrow 2 \geq x_1$

x_0: $\quad x_0 = \frac{4}{3} - x_1 \geq 0 \Rightarrow \frac{4}{3} \geq x_1$ ← tightest bound

change x_0, x_1

$x_0 - \frac{4}{3} - \frac{2}{3} w_1 - \frac{1}{3} w_2 = -x_1$ ← sub back

$x_1 = \frac{4}{3} + \frac{2}{3} w_1 + \frac{1}{3} w_2 - x_0$

$x_2 = \frac{1}{3} - \frac{1}{3} w_1 + \frac{1}{3} w_2$

$w_3 = \frac{2}{3} + \frac{1}{3} w_1 - \frac{1}{3} w_2 + x_0$

$z = \cancel{-\frac{4}{3}} + (\cancel{\frac{4}{3}} + \cancel{\frac{2}{3} w_1} + \cancel{\frac{1}{3} w_2} - x_0) - \cancel{\frac{2}{3} w_1} - \cancel{\frac{w_2}{3}}$

$$z = -x_0$$

s.t $\quad x_1 = \frac{4}{3} + \frac{2}{3} w_1 + \frac{1}{3} w_2 - x_0$

$$w_3 = \frac{2}{3} + \frac{1}{3} w_1 - \frac{1}{3} w_2 + x_0$$

$$x_2 = \frac{1}{3} - \frac{1}{3} w_1 + \frac{1}{3} w_2$$

Set $w_1, w_2, x_0 = 0 \Rightarrow \left.\begin{aligned} x_1 &= 4/3 \\ w_3 &= \frac{2}{3} \\ x_2 &= 1/3 \end{aligned}\right\}$ optimal !

Back to OG Problem

$$\max z = -2x_1 - x_2$$

s.t $\quad -x_1 + x_2 \leq 1 \qquad x_1, x_2 \geq 0$

$$-x_1 - 2x_2 \leq -2$$

$$x_2 \leq 1$$

$$z = -2\left(\frac{4}{3} + \frac{2}{3} w_1 + \frac{1}{3} w_2\right) - \left(\frac{1}{3} - \frac{1}{3} w_1 + \frac{1}{3} w_2\right)$$

$= -3 - w_1 - w_2$ and $w_1 = w_2 = 0$ so already optimal