CS47100: Introduction to AI Fall 2025

Assignment 3

NAME HERE Date: November 17, 2025

Problem 1 (a)

$E(aX) = aE(X) = a \cdot p$ by linear transformation

$E[(bX)]^2 = b^2 E[X^2] = b^2 \cdot E[X^2] = b^2 \cdot E[X] = b^2 \cdot p$

so, $\text{Var}(bX) = E[(bX)^2] - (E(bX))^2$

$\text{Var}(bX) = b^2 p - b^2 p^2 = b^2 p(1 - p)$

Problem 1 (b)

$E(X + Y) = E(X) + E(Y) = p + q$

$\text{Var}(X + Y) = Var(X) + Var(Y) + 2\text{Cov(X,Y)}$

Since, $X \perp\!\!\!\perp Y \Rightarrow \text{Cov}(X, Y) = 0$

$\text{Var}(X + Y) = p(1 - p) + q(1 - q)$

Problem 2 (a)

i Note that for fixed y with $P(Y = y) > 0$, the conditional distribution over X satisfies, $\sum_x P(X = x|Y = y) = 1$ because the probabilities of all mutually exclusive outcomes of X given $Y = y$ must sum to 1

So this expression is always 1 for any valid y.

ii Suppose $Y \in \{0, 1\}$ with $(P(Y = 0), P(Y = 1) > 0$

Then, $\sum_x \sum_y P(X = x|Y = y) = 2$

iii Suppose that $X, Y \in \{0, 1\}$ and $P(X = 1|Y = 0) = \frac{1}{2}$ and $P(X = 1|Y = 1) = \frac{1}{2}$

so, $\sum_y P(X = 1|Y = y) = \frac{1}{2} + \frac{1}{2} = 1$

but if we change to $P(X = 1|Y = 0) = \frac{3}{4}$, $P(X = 1|Y = 1) = \frac{3}{4}$

then, $\sum_y P(X = 1|Y = y) = \frac{3}{4} + \frac{3}{4} = \frac{3}{2} \neq 1$

Problem 2 (b) i

Suppose $\pi = P(S = 1)$ be the unknown

then $P(S = 1|M = 1) = \dfrac{P(M = 1|S = 1)P(S = 1)}{P(M = 1|S = 1)P(S = 1) + P(M = 1|S = 0)P(S = 0)}$

so, $P(S = 1|M = 1) = \dfrac{0.85\pi}{0.85\pi + 0.10(1 - \pi}$

so it depends on π, we cannot determine a value without base rate.

Next, $P(S = 0|M = 0) = \dfrac{P(M = 0|S = 0)P(S = 0)}{P(M = 0|S = 0)P(S = 0) + P(M = 0|S = 1)P(S = 1)}$

so, $P(S = 0|M = 0) = \dfrac{0.90(1 - \pi)}{0.90(1 - \pi) + 0.15\pi}$

again this depends on π so we cannot get a value without knowing the base rate. For both cases we can only express the value in terms of π.

Problem 2 (b) ii

Suppose $P(S = 1) = 0.20$
so, $\pi = 0.2$, $P(S = 0) = 0.8$

so, $P(S = 1|M = 1) = \dfrac{P(M = 1|S = 1)P(S = 1)}{P(M = 1|S = 1)P(S = 1) + P(M = 1|S = 0)P(S = 0)}$
$= \dfrac{0.85 \times 0.20}{0.85 \times 0.20 + 0.10 \times 0.80} = \dfrac{0.17}{0.25} = 0.68$

$P(S = 0|M = 1) = 1 - 0.68 = 0.32$

so a message moved to the Spam folder is more likely to be spam with .68 than legitimate .32.

Problem 3 (a)

i Since we cannot compute the joint $P(X_1, X_2|Y)$ or $P(X_1, X_2, Y)$. Thus we cannot determine what $P(Y|X_1, X_2)$ is, because multiple joint distributions are consistent with the same marginals.

ii $P(Y|X_1, X_2) = \dfrac{P(X_1, X_2|Y)P(Y)}{P(X_1, X_2)}$

iii We cannot determine it because we can't determine the value of $P(Y)$

iv Same case as (i) (multiple...)

Problem 3 (b)

i by conditional independence we know $P(X_1, X_2|Y) = P(X_1|Y)P(X_2|Y)$ to form

$P(X_1, X_2, Y) = P(X_1|Y)P(X_2|Y)P(Y)$

if we fixed x_1, x_2 we get $P(Y = y|x_1, x_2) = \dfrac{P(x_1|y)P(x_2|y)P(y)}{\sum_y' P(x_1|y')P(x_2|y')P(y')}$

now it is sufficient.

ii Already sufficient.

iii Not sufficient because we are still missing $P(Y)$

iv By conditional independence we can construct $P(X_1, X_2|Y) = P(X_1|Y)P(X_2|Y)$ and then it is them same as original case (ii), thus it is sufficient now

Problem 4(a)

Node C has Parent A, so the CPT is $P(C|A)$

$|Dom(A)| = |Dom(C)| = 3 \Rightarrow$ Number of entries in the CPT for C = 9.

Problem 4(b)

All undirected paths from D to E: $D \to F \leftarrow E$

Note at F we have a collider on this path. So we are not conditioning on F so this path is blocked. So, there is no other path connecting D and E in the graph. So $D \perp\!\!\!\perp E$ true.

Problem 4(c)

Consider path between B and F

path $B \to D \to F$

Along B - D - F, at D we have a chain $B \to D \to F$. We are conditioning only on C, not on D. So this path is open.

Any other path must go to through this same structure, so at least one open path exists.

Thus B and F are not d-separated given C $\Rightarrow B \perp\!\!\!\perp F|C$ is false

Problem 4(d)

Paths between A and B:

direct path $A \to B$

this is a simple chain with non collider B. We are conditioning only on F, not A or B, so this path remains opens.

Thus we conclude that A and B are d-connected given F

$A \perp\!\!\!\perp B|F$ is false

Problem 4(e)

Path $A \rightarrow B \rightarrow D \rightarrow F \leftarrow E$

B appears as $A \rightarrow B \rightarrow D$ so B is non collider.

D is non collider along this path and we are conditioning on D, thus the only path $A - ... - E$ is blocked by conditioning on D, and there is no other path.

$A \perp\!\!\!\perp E|D$ is true

Problem 5 (a)

$B \to A, W \to A, C \to A, X \to A, A \to S, X \to S$

So A has parents $\{B, W, C, X\}$ and S has parents $\{A, X\}$.

All of B, W, C, X are root nodes.

Problem 5 (b)

$P(B, W, C, X, A, S) = P(B)P(W)P(C)P(X)P(A|B, W, C, X)P(S|A, X)$

Problem 5 (c)

inexperienced as $X = 0$, battery healthy as $B = 1$, weather favorable as $W = 1$, sensors well calibrated as $C = 1$, mission success as $S = 1$

$P(S = 1|B = 1, W = 1, C = 1, X = 0) = \frac{\sum_a P(B=1,W=1,C=1,X=0,A=a,S=1}{\sum_s \sum_a P(B=1,W=1,C=1,X=0,A=a,S=s)}$
$= \frac{\sum_a P(B=1)P(W=1)P(C=1)P(X=0), P(A=a|1,1,1,0)P(S=1|a,0)}{\sum_s \sum_a P(B=1)P(W=1)P(C=1)P(X=0)P(A=a|1,1,1,0)P(S=s|a,0)}$
$= \frac{\sum_a P(A=a|1,1,1,0)P(S=1|a,0)}{\sum_s \sum_a P(A=a|1,1,1,0)P(S=s|a,0)}$
for each fixed $(a, 0), \sum_s P(S = s|a, 0) = 1$, and the denominator becomes 1

so, $P(S = 1|B = 1, W = 1, C = 1, X = 0) = \sum_{a \in \{0,1\} P(A=a|B=1,W=1,C=1,X=0) P(S=1|A=a,X=0)}$

Problem 5 (d)

$P(B = 1 \mid S = 1, W = 0, C = 1, X = 1)$ By Bn factorization

P(B,W,C,X,A,S) = P(B)P(W)P(C)P(X)P(A| $B, W, C, X)P(S \mid A, X)$,

and summing out A, we get $P(B = 1 \mid S = 1, W = 0, C = 1, X = 1) = \frac{P(B=1,S=1,W=0,C=1,X=1)}{\sum_{b\in\{0,1\}} P(B=b,S=1,W=0,C=1,X=1)}$, with

$$P(B = b, S = 1, W = 0, C = 1, X = 1) = \sum_{a\in\{0,1\}} P(B = b)P(W = 0)P(C = 1)P(X = 1)P(A = a \mid b, 0, 1, 1)P(S = 1 \mid a, 1).$$

The common factor $P(W=0)P(C=1)P(X=1)$ cancels, so define

$$\text{score}(b) = P(B=b)\sum_a P(A=a \mid b,0,1,1)P(S=1 \mid a,1),$$

and then

$$P(B=1 \mid S=1, W=0, C=1, X=1) = \frac{\text{score}(1)}{\text{score}(0)+\text{score}(1)}.$$

From the CPTs:

$$\begin{aligned}
&P(B=1)=0.50, \quad P(B=0)=0.50,\\
&P(A=1 \mid B=1, W=0, C=1, X=1)=0.70, \quad P(A=0 \mid 1,0,1,1)=0.30,\\
&P(A=1 \mid B=0, W=0, C=1, X=1)=0.55, \quad P(A=0 \mid 0,0,1,1)=0.45,\\
&P(S=1 \mid A=0, X=1)=0.55, \quad P(S=1 \mid A=1, X=1)=0.90.
\end{aligned}$$

Hence

$$\text{score}(1) = 0.50\big(0.30\cdot 0.55 + 0.70\cdot 0.90\big) = 0.50\cdot 0.795 = 0.3975,$$

$$\text{score}(0) = 0.50\big(0.45\cdot 0.55 + 0.55\cdot 0.90\big) = 0.50\cdot 0.7425 = 0.37125.$$

Therefore

$$P(B=1 \mid S=1, W=0, C=1, X=1) = \frac{0.3975}{0.3975+0.37125} = \frac{106}{205} \approx 0.517.$$

So the battery is more likely to have been healthy than unhealthy.